THE

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EDITED BY

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THE MATHEMATICAL ASSOCIATION

THE Annual General Meeting of the Mathematical Association was held at the University of Sheffield on 8th, 9th, 10th, 11th April, 1953.

On 8th April members were welcomed by the Vice-Chancellor, Dr. J. M. Whittaker, at a reception held in the Firth Hall.

On 9th April, the business meeting was held at 9.15 a.m., with the President, Mr. K. S. Snell, in the chair. The Report of the Council for 1952 was adopted, and a financial statement given by the Treasurer was received.

The election of Professor T. A. A. Broadbent as President for 1953 was announced. The existing Vice-presidents were re-elected, and Mr. K. S. Snell became a Vice-president under Rule 19. The Treasurer, the Secretaries, the Editor of the *Mathematical Gazette*, and the Auditor were re-elected. The following were elected to serve on the Council: Prof. T. Arnold Brown, Dr. I. W. Busbridge, Prof. R. L. Goodstein, Mr. W. J. Langford, Dr. E. A. Maxwell, Mr. F. J. Phillips, Mr. M. A. Porter, Mr. A. P. Rollett.

At 9.45 a.m. Mr. K. S. Snell gave his Presidential address, " School mathematics today and tomorrow ". At 11.30 a.m., Professor A. G. Quarrell spoke on "Metallurgy—craft and science ".

In the afternoon, members visited places of interest in the Sheffield area, and theatre parties were arranged for the evening.

On 10th April, at 9.15 a.m., a discussion on " Infinity " was opened by Professor A. G. Walker and Dr. Christine Hamill; at 11.15 a.m., Dr. Fox gave a lecture on the work and equipment of the mathematics division of the National Physical Laboratory. At 2 p.m., a discussion, " From Primary School to Secondary School ", was opened by Mr. L. B. Birch, Mrs. E. M. Williams and Mr. M. W. Brown, and at 4.45 p.m. Dr. S. Vajda gave a paper on the " Theory of games ".

On 11th April, at 9.15 a.m., Mr. J. F. Hinsley gave a paper entitled " From examination question to industrial problem ".

A Publishers' Exhibition, an Exhibition of Teaching Aids, and an Exhibition of Calculating Machines were open during the meeting.

MATHEMATICS GRADUATE LIBRARY

REPORT OF THE COUNCIL FOR THE YEAR 1952

Membership.

During the year, 34 junior and 161 other members joined the Association. The number of members on the 1st November, 1952, was 2,506, this total being made up of 8 honorary members, 232 life members, 2,180 ordinary members and 86 junior members. The figures show an increase of 9 in ordinary membership during the year ; this is the first increase since the subscription was raised in 1949. It is with regret that the Council reports the death of the following members : Mr. G. Burtenshaw (1949), Mr. W. I. Cooke (1915), Dr. S. H. Cracknell (1945), Mr. T. R. Dawson (1938), Mr. D. C. Fraser (1926), Miss E. R. Gwatkin (1902), Miss E. M. Gwillim (1920), Mr. C. L. M. Jockel (1935), Mr. I. F. H. Jones (1920), Dr. R. R. Jones (1937), Mr. C. H. Richards (1898), Mr. S. Schofield (1951), Mr. J. E. Tarver (1925), Miss M. Taylor (1937), Rt. Rev. F. J. Western (1942), Mrs. L. E. Wilkinson (1908), Mr. C. E. Youngman (1898).

Miss Gwatkin was Honorary Secretary of the Association in 1910.

Finance.

Receipts during the year, including subscriptions in advance, were £3,876 12s. 10d., and payments were £3,695 18s. 8d. The balance of cash in hand increased by £180 14s. 2d. and on 1st November, 1952, stood at £758 13s. 11d.

Against the increase of cash in hand there must be placed two items : the amount due to the Association from the sale of Gazettes and Reports is £62 14s. 5d. lower than the previous year, while subscriptions paid in advance are £29 14s. 6d. higher. The effective increase in what is now called the General Fund is therefore not £180 14s. 2d., but £88 5s. 3d.

The seven-year covenants for annual subscribers have produced £350 11s. 0d., a new and substantial source of income. Approximately 400 members have signed the covenant. As a result the Teaching Committee should be able to go ahead with their plans for publication during 1953.

Prophecy in print is perhaps foolhardy, but it does seem likely that the working surplus from the next account, together with the cash now in hand, should be sufficient to finance publications by the Teaching Committee costing some £1,150.

The Branches Equalisation Fund has now been closed, but Council will continue to help by paying visiting lecturers' expenses, when such help is necessary. It is hoped that money paid to Branches in this way will be treated as a loan. The Treasurer has been authorised to make a payment, not exceeding £5, to assist new Branches during their period of formation.

The Mathematical Gazette.

The yearly content of about 300 pages is being maintained. Since the number of items submitted, and the number of books sent for review, both continue to increase, the queue of material in type grows longer, in spite of the application of harsher standards for acceptance. Many more articles are being submitted from abroad, but only a small proportion can be used. In the interests of economy, plates will appear on an average only twice a volume.

The Library.

The lending service has been working throughout this year, about seventy books having been asked for and sent out by the librarian at Reading University, where the library is at present housed. Unfortunately a change has to

be made. When Reading University offered to house the library in 1947 it was expected that they would shortly have new buildings in which their library could be expanded. The buildings have not materialised and are unlikely to do so in the near future. There is consequently not adequate accommodation for the library of the Association, and a transfer has been arranged to the library of the University College of Leicester, where there are new buildings and good accommodation. Council is grateful to the authorities at Leicester for making this offer, and it is hoped that the books will be transferred shortly and that the lending service will not seriously be interrupted. Council wishes to place on record its gratitude to Reading University, and in particular to the librarian of the University, for their help over the last six years.

The Teaching Committee.

The Committee met in London on 5th January, 1952. A draft report on *The Teaching of Higher Geometry in Schools* was considered, and adopted subject to slight modifications and to the drafting of a preamble for approval by the Teaching Committee. The preamble was subsequently circulated, and the resulting comments incorporated and agreed upon between the Chairman of the Teaching Committee and the Secretary of the Geometry subcommittee. The Report has now reached the proof stage, and is to be published during 1953 (at 7s. 6d. to non-members).

A draft report on Visual Methods was also considered, and was referred to an *ad hoc* subcommittee with instructions to produce a draft confined to direct applications of Visual Aids to the teaching of mathematics. This subcommittee, after consulting the chairmen of allied subcommittees, will be reporting soon.

Other draft reports which the Teaching Committee hopes to consider in April are one from the Primary Schools subcommittee and one on the teaching of mathematics in Technical Colleges. Work on a report on the use of the history of mathematics in teaching is in abeyance, but should be resumed in 1954.

A revised *List of Books suitable for School Libraries* has been prepared, and the approval of the Teaching Committee for its publication will also be sought in April.

The Standing Subcommittee of the Teaching Committee has dealt with a few complaints about errors in examination papers, with a request from the Civil Service Commissioners for comments on a revised syllabus for their examinations for the Executive Class, and with an enquiry from the Association of University Teachers for an opinion on alleged over-specialisation in sixth forms in schools. It has also taken note of a demand for a re-issue of the Jeffery Report which recommended alternative syllabuses at School Certificate examinations, and is to recommend the Association to reprint this Report, subject to the approval of all concerned.*

The existing Reports of the Association continue to sell well; those on Arithmetic, Algebra and Trigonometry have all been reprinted recently; there is no reason to increase the prices at which they will be sold.

Mr. L. P. Dealtry has been elected to the Teaching Committee in place of Mr. G. Ashton (resigned). The Committee received with much regret the resignation of Mr. J. C. Manisty from the post of Secretary, in which position he had contributed much to the work now reaching completion; Mr. M. W. Brown has succeeded him, and has been replaced as Assistant Secretary by Mr. B. J. F. Dorrington.

* The reprint is circulated with this issue of the *Gazette*.

The Association was well represented by members of the Teaching Committee and other members at an all-day conference on Mathematics in Secondary Modern Schools organised in November by the University of London Institute of Education and attended by about 500 people.

The Branches.

Branches at home and overseas continue to flourish and reports of their activities appear from time to time in the *Mathematical Gazette*. Nottingham Branch is being revived and new Branches have been, or are being, formed in East Africa and Malaya.

An interesting development at home is the sponsorship by members in the Midlands of a Junior Mathematical Society, which meets at least once per term and which is attracting support from many schools.

Problem Bureau.

Once again the number of applications received has increased, many coming from members who are applying for the first time. Applicants are consistently appreciative of the help they receive. The Bureau is under the direction of Mr. A. S. Gosset Tanner, who asks, and Council wholeheartedly agrees, that the willing and competent assistance he has from his " staff " should be again publicly acknowledged. These members are Messrs. R. H. Cobb, G. A. Garreau, R. V. H. Roseveare, W. J. Hodgetts, G. A. Bull and W. Hope-Jones.

The Association is also deeply indebted to Mr. Gosset Tanner.

Officers and Council.

Council wishes to offer, on behalf of all members of the Association, its sincere thanks to the President, Mr. K. S. Snell, and to the Officers for the work which they have done during the year. Miss F. M. A. Pendry and Mr. D. G. Bousfield will be retiring from the Council, and the Association's thanks are due to them also for their services.

GLEANINGS FAR AND NEAR

1736. At Redbrick they treat mathematics as an instrument of technology; at Cambridge they regard it as an ally of physics and an approach to philosophy; at Oxford they think of it as an art in itself having affinities with counterpoint and dancing. At any rate it is the great intermediary.—Music critic of *The Times*, May 2, 1952. [Per Prof. H. Heilbronn, Dr. I. W. Busbridge.]

1737. Two Sides to Every What?

To every piece of paper of course. Even though the mathematicians will explain to you that there are six (they count the edges) you don't have to be a patriot, a non-waste-paper enthusiast, to write on more than two.—*Everybody's Weekly*, March 29, 1952. [Per Mr. H. G. Woyda.]

1738. The authors point out that at present out of every 100 tons of coal consumed 80 are wastefully dispersed through inefficient heating and generating systems.

Technologists agree that with scientific direction these proportions could be almost completely reversed.—*Daily Mail*, December 18, 1951. [Per Mr. R. A. Fairthorne.]

1739. The article " Throne and Church " in your issue of February 15 is marred by a statement which is neither more nor less than a half-truth.—Letter to *The Times*, February 20, 1952. [Per Mr. C. E. Kemp.]

A PROPERTY OF LINEAR CYCLIC TRANSFORMATIONS.

BY J. H. CADWELL.

1. *Summary.*

The mid-points of the sides of any quadrilateral form a parallelogram. Repeated applications of this inscription process will lead to two families of similar and similarly situated figures. It is proved below that the general n-sided polygon lying in space of h dimensions gives rise to similar sequences in the limit. If the initial polygon is not re-entrant convergence is rapid, as will be seen from Figures 1 and 5. The parallelogram is replaced by the affine transformation of a regular polygon lying in a plane.

We denote the rth vertex of the xth polygon P_x, by the vector z_r^x. Then the process is represented by

$$2z_r^{x+1} = z_r^x + z_{r+1}^x.$$

Another simple construction leads to

$$2z_r^{x+1} = z_{r-1}^x + z_{r+1}^x.$$

This produces a similar situation, but the limiting forms are now crossed, and sometimes degenerate, polygons. This is so provided P_0 is not a regular simple polygon. Speed of convergence to the limiting form depends on the degree of departure of the original from the simple regular form. In Figures 3 and 4, re-entrant originals were used to ensure speedy convergence. The black polygons shown are the results of four applications of the process.

The construction implied by

$$3z_r^{x+1} = 2z_r^x + z_{r+1}^x$$

results in more complex behaviour ; the limiting sequences do not appear. However, there is still a limiting recurrence property ; and in the limit the polygons are still affine transformations of a regular polygon.

Below we consider the general transformation.

$$z_r^{x+1} = \sum_{s=0}^{n-1} \alpha_s z_{r+s}^x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

It will be shown that the behaviour of the trisection construction mentioned above is typical. The two bisection constructions correspond to a simplification arising when certain quantities are rational numbers.

2. *Cyclic matrices.*

A square matrix is called cyclic if its pqth element is α_{q-p} where $\alpha_{n+x} = \alpha_x$. Thus

$$A = \{\alpha_{q-p}\} = \begin{pmatrix} \alpha_0, & \alpha_1, \ldots, & \alpha_{n-1} \\ \alpha_{n-1}, & \alpha_0, \ldots, & \alpha_{n-2} \\ \vdots & & \\ \alpha_1, & \alpha_2, \ldots\ldots, & \alpha_0 \end{pmatrix}$$

Consider first the case of odd n. By means of Fourier's expansion the coefficients can be expressed in the form

$$\alpha_j = \frac{\lambda_0}{n} + \frac{2}{n} \sum_{k=1}^{(n-1)/2} \lambda_k \cos(2\pi kj/n - \phi_k),$$

$$\lambda_k \exp(i\phi_k) = \Sigma \alpha_s \exp(2\pi iks/n).$$

Here and throughout this note, the summation sign when unaccompanied by limits indicates summation with regard to s for the values $0, 1, 2, \ldots, (n-1)$.

The matrix can now be expressed in the form

$$\Lambda = \lambda_0 C_0 + \sum_{k=1}^{(n-1)/2} \lambda_k \{\exp(-i\phi_k) \,.\, C_k + \exp(i\phi_k) \,.\, C_{-k}\}$$
$$= [\lambda_k, \phi_k].$$

The C-matrices are defined by their pqth elements as

$$C_k = \{n^{-1} \exp[2\pi i k(q-p)/n]\}.$$

These matrices are readily shown to obey the multiplicative rule

$$C_x \,.\, C_y = \delta_{xy} \,.\, C_x.$$

Thus if Λ and Γ are cyclic and of the same order, and if l is a positive integer, the results

$$\Lambda\Gamma = [\lambda_k \gamma_k, (\phi_k + \psi_k)],$$
$$\Lambda^l = [\lambda_k{}^l, l\phi_k]$$

follow at once. The latter result holds for negative l provided that none of the λ is zero. When n is even a similar result follows readily.

3. *Sequences of polygons.*

If the quantities z_r^s for $r = 0, 1, \ldots, (n-1)$ are regarded as the elements of an $n \times 1$ matrix $Z^{(s)}$, then (I) becomes

$$Z^{(s+1)} = \Lambda Z^{(s)}.$$

The result of l applications of (I) followed by the transformation

$$w_r^l = z_{r-u}^l$$

gives the following expression when n is odd :

$$w_r^l = \frac{1}{n} \lambda_0^l \Sigma z_s^0 + \frac{2}{n} \sum_{k=1}^{(n-1)/2} \lambda_k^l \Sigma z_s^0 \cos\{2\pi k(s-r+u)/n - l\phi_k\}. \quad \ldots\ldots\ldots(II)$$

For even n the second summation is from 0 to $\frac{1}{2}(n-2)$ and there will be the additional term

$$n^{-1} \,.\, \lambda_{\frac{1}{2}n}^l \Sigma z_s^0 \cos \pi(s-r+u).$$

If the quantities $|\lambda_k|$ have a unique maximum $|\lambda_m|$ $(m \neq 0)$, then for large l,

$$w_r^l \simeq (2/n) \,.\, \lambda_m^l \Sigma z_s^0 \cos\{2\pi m(s-r+u)/n - l\phi_m\}. \quad \ldots\ldots\ldots\ldots(III)$$

If $|\lambda_0|$ is the maximum modulus, let $|\lambda_m|$ be the greatest of the remaining terms. Then

$$w_r^l \simeq \frac{1}{n} \lambda_0^l \Sigma z_s^0 + \frac{2}{n} \lambda_m^l \Sigma z_s^0 \cos\{2\pi m(s-r+u)/n - l\phi_m\}. \quad \ldots\ldots\ldots(IV)$$

The second summation in this expression can be written as

$$\cos\psi_r \Sigma z_s^0 \cos(2\pi m s/n) + \sin\psi_r \Sigma z_s^0 \sin(2\pi m s/n), \quad \ldots\ldots\ldots\ldots(V)$$

where the angles ψ_r are defined by

$$\psi_r = 2\pi m(r-u)/n + l\phi_m. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(VI)$$

The polygon defined by the right-hand side of (III) or (IV) will have the same shape and orientation as the polygon defined by (V). This polygon lies in a plane, and its vertices are linked by an affine transformation to the points of the unit circle defined by the angles ψ_r for $r = 0, 1, 2, \ldots, (n-1)$. If m and n are co-prime these points form a regular polygon of n sides, star-shaped unless $m = 1$. If $(m, n) = d$ they form a polygon with n/d vertices each counted d times.

4. *Recurrence properties.*

First consider the case when the quantity ϵ defined below is rational and let

$$\epsilon = n\phi_m/2\pi m = a/b, \quad (2m, b) = f, \quad p = b(n, f)/f$$

where a and b are integers. Then it can be shown that when l is increased by p the angles defined by (VI) either recur or are increased by π radians. Moreover, p is the smallest integer with this property.

Thus the approximating polygons (III) or (IV) will fall into p families each of the same shape, and with successive members of a family differing in orientation by zero or π radians. Consequently, in the limit the sequence $P_0, P_1, \ldots$ will behave in the same way. Illustrations of this type of behaviour are given in the next paragraph.

If ϵ is irrational, it will be possible to find successively closer rational approximations to it. Thus the polygon (III) or (IV) will reappear with successively better degrees of approximation; the same will be true of the sequence $P_0, P_1, \ldots$ in the limit.

The case

$$3z_r^{x+1} = 2z_r^x + x_{r+1}^x$$

is illustrated by Figure 2. Here $m = 1$ and

$$\epsilon = \frac{1}{3+}\,\frac{1}{7+}\,\frac{1}{7+}\cdots.$$

It will be found that a polygon, far enough on in the sequence for the approximation (III) to be used, will recur in practically the same shape and position after 3, 22, 157, ... operations. The heavily drawn hexagons in the figure agree quite well in both shape and orientation.

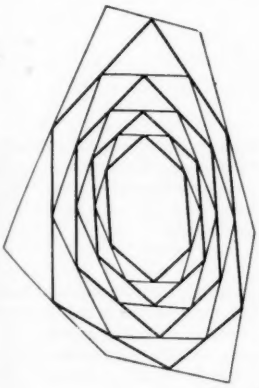


FIG. 1

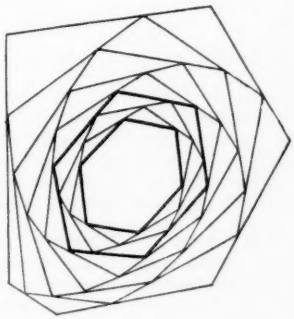


FIG. 2

5. *Symmetrical transformations.*

If the coefficients are symmetrical with regard to one of their number, *i.e.* if

$$\alpha_{t-s} = \alpha_{t+s},$$

it will be found that $p = 1$, and in the limit there is one family. The case

$$2z_r^{x+1} = z_{r-1}^x + z_{r+1}^x$$

corresponds to a simple geometrical construction and in Figure 3, a pentagon together with the results of three and four applications of the transformation are shown. The dominant term is $m=2$, and the limiting shape is that of a star pentagon. It will be seen that successive members differ in orientation by π radians. In Figure 4, $n=6$, and $m=3$ is the dominant term. The limiting shape consists of a " line " polygon counted three times. The one shown in the figure in black is the result of four operations.

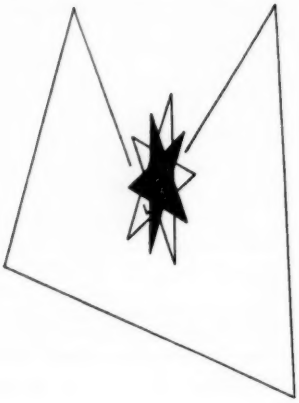


FIG. 3

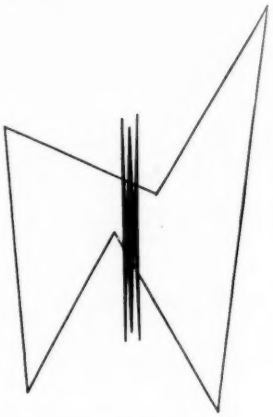


FIG. 4

If there is symmetry with regard to a pair of coefficients, *i.e.* if

$$\alpha_{t-s}=\alpha_{t+s+1},$$

it will be found that, for even n, $p=2$ and two distinct families arise. For odd n, $p=1$ and there is a change of orientation of π radians between successive members. The simple case of bisection of sides is of this type, and the results for even and odd n appear in Figures 1 and 5 respectively. As $m=1$ is the dominant term, the limiting polygons are always simple.

6. *Some further considerations.*

For special distributions of the vertices of the original polygon the factor multiplying the dominant term may vanish. Thus if the original is a simple regular polygon all terms except the first two vanish, and the result of any cyclic transformation is a simple regular polygon. For an irregular convex polygon, the multipliers of the terms $\lambda_2, \lambda_3, \ldots$ will be small compared with that of λ_1. Thus for a transformation with $m=1$ the disturbing effect of the other terms will be small. The rapid convergence in Figures 1 and 5 arises from this cause. In Figures 3 and 4, where $m=2$, a re-entrant original was used in order to make convergence reasonably fast.

If there are two values $\lambda_m=\lambda_m'$ greater in modulus than the rest the limiting form will be four dimensional (unless $h<4$). The sequence will still show the same general behaviour ; although for definite limiting families two quantities will have to be rational. If there are two dominant terms of equal magnitude but opposite in sign, there will be distinct forms for odd and even n.

The case

$$(p+q)z_r^{x+1} = pz_r^x + qz_{r+v}^x \quad (n = 2v)$$

is illustrated in Figure 6. Here the result of three operations is shown with $p:q = 3:7$, the dotted triangle indicates the limiting form in both shape and position. In this case, all even λ are unity and all odd λ are $(p-q)/(p+q)$.

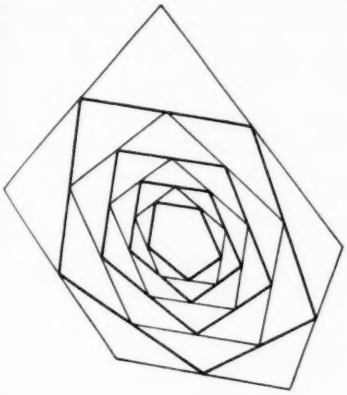


FIG. 5

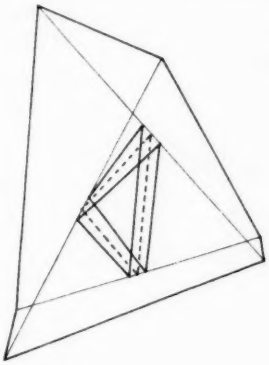


FIG. 6

Convergence is rapid if $(p-q)$ is small; in the case $p=q$ the limiting shape appears at the first operation and remains unaltered thereafter. This limiting shape will be v dimensional provided $h > v$.

J. H. C.

1740. PROFESSORS OF APPLIED MATHEMATICS.—"Are you a mathematician?" "Not so as you'd notice it." "Neither am I. I don't know much about the pure mathematics of the theory of probabilities, but I know a little about the application of it." "What do you mean?" "Well, you know the professors of applied mathematics who disguise themselves as bookies? They will give you odds of twenty to one on an outsider, won't they?" "Yes, sometimes." "That means that in return for the pleasure of holding your modest five bob for ten minutes or so they will give you five pounds out of the goodness of their hearts and the fulness of their pockets in the unlikely event of your horse winning. In actual fact it loses and your five bob helps them to pay their super-tax; but that doesn't affect our argument. On the next race they will give you ten to one on another horse that doesn't stand much chance. But suppose you back these two horses for a double; that is, you only win if both your selections come in first. Do they add the odds together and give you the thirty to one? No; such is their benevolence that they multiply them, and your five bob will win you fifty pounds instead of seven pounds ten."—*The Missing Money-Lender*, by W. S. Sykes, Chap. 9 (1931). [Per Prof. E. H. Neville.]

1741. The scientist, pure or applied, is still often treated as an uncouth Philistine. I find this a perverse charge, coming from men whose cultural interests fit primly into the clues to *The Times* crossword.—Dr. J. Bronowski, *The Observer*, April 22, 1951. [Per Mr. P. Vernes.]

RECIPROCAL NOMOGRAMS.

By C. V. Gregg.

The reciprocal nomogram described in Note 2108 (February, 1950) for the solution of $1/R = 1/R_1 + 1/R_2$ is a particular case of a more general property.

Theorem. If A, B, C are three collinear points and O a point not on ABC and any line cut OA, OB, OC in X, Y, Z (Fig. 1), then, paying regard to sign,

$$\frac{\sin BOC}{OX} + \frac{\sin COA}{OY} + \frac{\sin AOB}{OZ} = 0, \quad \text{.....(i)}$$

and

$$BC \cdot \frac{XA}{OX} + CA \cdot \frac{YB}{OY} + AB \cdot \frac{ZC}{OZ} = 0. \quad \text{.....(ii)}$$

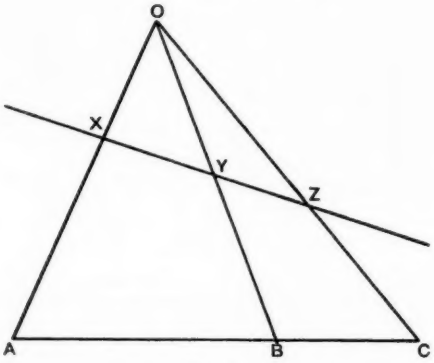


Fig. 1.

For,

$$\triangle OBC + \triangle OCA + \triangle OAB = 0,$$

so that

$$\tfrac{1}{2}OB \cdot OC \sin BOC + \tfrac{1}{2}OC \cdot OA \sin COA + \tfrac{1}{2}OA \cdot OB \sin AOB = 0,$$

and hence

$$\frac{\sin BOC}{OA} + \frac{\sin COA}{OB} + \frac{\sin AOB}{OC} = 0.$$

Similarly

$$\frac{\sin BOC}{OX} + \frac{\sin COA}{OY} + \frac{\sin AOB}{OZ} = 0. \quad \text{.....(i)}$$

Subtracting,

$$\sin BOC \cdot \frac{OA - OX}{OA \cdot OX} + \sin COA \cdot \frac{OB - OY}{OB \cdot OY} + \sin AOB \cdot \frac{OC - OZ}{OC \cdot OZ} = 0,$$

and thus

$$\frac{\sin BOC}{OA} \cdot \frac{XA}{OX} + \frac{\sin COA}{OB} \cdot \frac{YB}{OY} + \frac{\sin AOB}{OC} \cdot \frac{ZC}{OZ} = 0.$$

Now

$$\frac{\sin BOC}{OA} : \frac{\sin COA}{OB} : \frac{\sin AOB}{OC}$$

$$= OB \,.\, OC \sin BOC : OC \,.\, OA \sin COA : OA \,.\, OB \sin AOB$$
$$= \triangle OBC : \triangle OCA : \triangle OAB$$
$$= BC : CA : AB,$$

whence

$$BC \,.\, \frac{XA}{OX} + CA \,.\, \frac{YB}{OY} + AB \,.\, \frac{ZC}{OZ} = 0. \quad \text{.....(ii)}$$

An alternative proof of (ii) is now given, and (i) can be regarded as the particular case of (ii) in which ABC is the line at infinity.

$$\triangle OYZ : \triangle OBC = OY \,.\, OZ : OB \,.\, OC.$$

But

$$\triangle OYZ + \triangle OZX + \triangle OXY = 0.$$

Hence

$$\triangle OBC \,.\, \frac{OY \,.\, OZ}{OB \,.\, OC} + \triangle OCA \,.\, \frac{OZ \,.\, OX}{OC \,.\, OA} + \triangle OAB \,.\, \frac{OX \,.\, OY}{OA \,.\, OB} = 0,$$

so that

$$\triangle OBC \,.\, \frac{OA}{OX} + \triangle OCA \,.\, \frac{OB}{OY} + \triangle OAB \,.\, \frac{OC}{OZ} = 0.$$

But

$$\triangle OBC : \triangle OCA : \triangle OAB = BC : CA : AB,$$

and thus

$$BC \,.\, \frac{OA}{OX} + CA \,.\, \frac{OB}{OY} + AB \,.\, \frac{OC}{OZ} = 0.$$

Therefore, since

$$BC + CA + AB = 0,$$
$$BC \frac{XA}{OX} + CA \,.\, \frac{YB}{OY} + AB \,.\, \frac{ZC}{OZ} = 0.$$

Corollary 1. If C be at infinity, (ii) reduces to

$$XA/OX - YB/OY + AB/OZ = 0. \quad \text{.....(iii)}$$

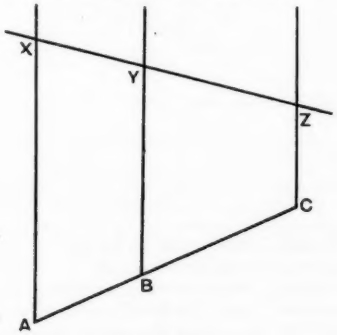


FIG. 2.

Corollary 2. If O be at infinity, (ii) reduces to the "three parallels property"

$$BC \,.\, AX + CA \,.\, BY + AB \,.\, CZ = 0. \quad \text{.....(iv)}$$

This is the basis of most logarithmic nomograms (Fig. 2).

In the particular case when $\angle AOB = \angle BOC = 60°$, (i) reduces to

$$1/OX - 1/OY + 1/OZ = 0.$$

This is the case used in the nomograms of Note 2108 (Fig. 3).

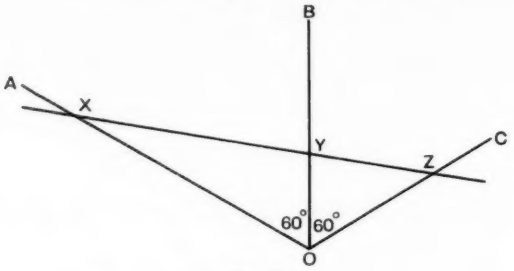


FIG. 3.

With proportional scales (ii) may be used for nomograms $lx + my + nz = 0$ and for

$$l/x + m/y + n/z = 0$$

with a range of values of the variables from zero to infinity.

(A line OA is graduated with a proportional scale when the number associated with a particular point P on it is a given multiple of the ratio $OP : PA$. Such a scale is most easily constructed as the projection of a linear scale, as shown in Fig. 4, where $OP/PA = (1/AV) \,.\, OL$).

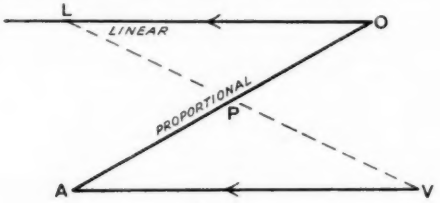


FIG. 4.

The construction of a reciprocal nomogram with linear scales.

The labour of dividing the scales may be avoided by using a piece of squared paper and making $\angle AOC$ a right angle. The outer scales may be equal, but a more useful nomogram is obtained by using unequal scales. The method of marking off is, I think, clearly shown in Fig. 5, where the dotted lines represent the printed lines of the graph paper.

If the nomogram is to be constructed on drawing paper, OB, OC may be drawn at any convenient angle, preferably acute, and marked off in any desired linear scales. OB is drawn along the diagonal of a parallelogram, three of whose vertices are O and two points of OA, OC representing the same quantity, and is graduated by drawing through the graduations of OC lines parallel to OA.

It is worth noting that if a number of values of u, v for a lens or mirror have been measured, the consistency of the results can be checked and the value of f found by joining $(u, 0)$, $(0, v)$ for each pair of values. These lines

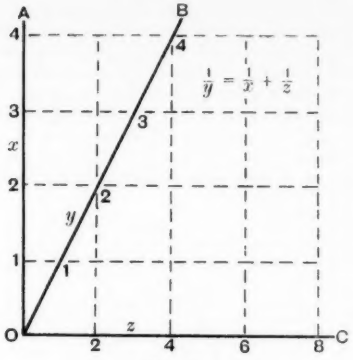


FIG. 5.

(assuming that the convention used makes $1/u + 1/v = 1/f$) should concur in the point (f, f) as indicated in Fig. 6.

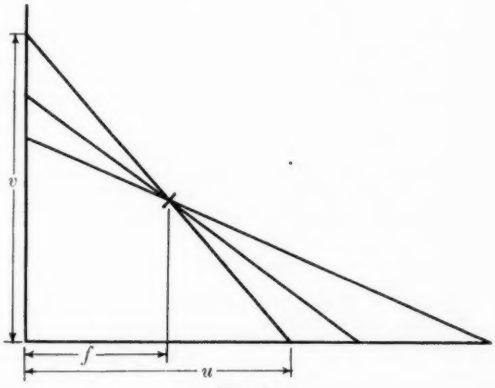


FIG. 6.

The construction of a reciprocal nomogram with non-linear scales.

Although the nomogram of Fig. 5 is useful and quickly constructed, it is more convenient to have one of the outer scales graduated from 0 to 10 and the other from 0 to ∞, as in Fig. 7. If blue prints are not required this also can be constructed on graph paper. It is a modification of Fig. 1 with A at infinity; OA is graduated with linear scale and OB, OC with *proportional* scales.

OA is laid off and divided uniformly. OC is drawn at any convenient angle and scaled from zero at O to infinity at C with any desired number n allotted to its midpoint. (In Fig. 7, $n = 10$.) OB passes through the midpoint of the line joining C to the nth graduation of OA. P is a point such

that CP is parallel to OA and equal to the distance from O to the nth graduation of OA. OC is graduated from 0 to 10 by projecting the scale of CA from P. It is then graduated from 10 to 20 by projecting the scale from Q, the midpoint of PC and doubling the associated number, and further graduated to 100 by projecting from R, such that CR is one-tenth of CP. OB may be graduated by projecting the scale of OA from P and halving the associated number, or by projecting the scale of OA from C, or by projecting the scale of OC by lines parallel to OA. Typical construction lines are shown dotted in Fig. 7. If preferred, both the outer scales may be proportional.

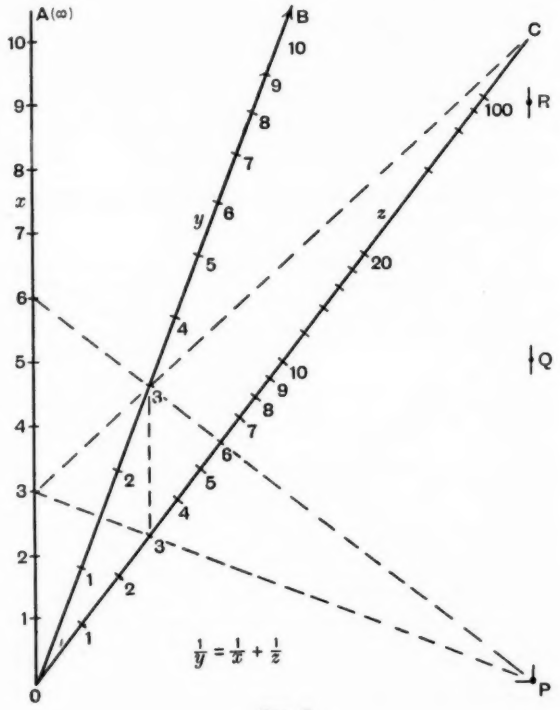


FIG. 7.

Reciprocal nomogram with curved scale.

The nomograms described represent the variables of the formula

$$1/x - 1/y + 1/z = 0$$

on three separate scales, but alternatively x and z can be represented on an arc of a conic (or circle) and y, linearly or proportionally, on a tangent to this conic. Such a nomogram, which is not very convenient to use, is shown in Fig. 8, where OA is the arc and T the pole of OA. The tangent OT is scaled

proportionally from zero at O to infinity at T, and OA is graduated by projecting this scale from A. A linear scale for y is obtained by making OA a

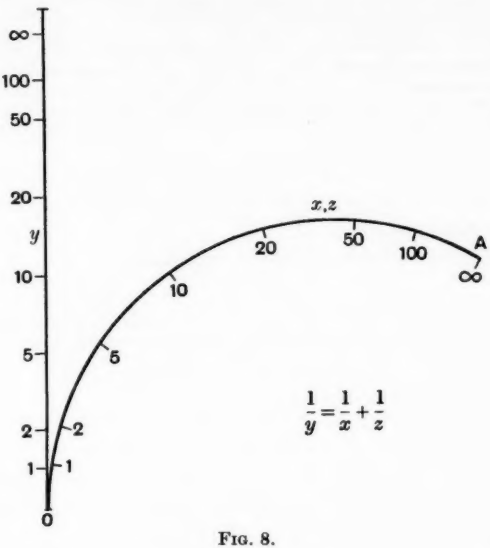


FIG. 8.

diameter or by making T the centre and TO an asymptote of a hyperbola. The conic property on which this nomogram depends is the basis of a useful nomogram for the solution of quadratic equations which may be described in a later note. C. V. G.

1742. This " jazzing " of marches was achieved partly by the trick of shifting the accent from the strong to the weak beat and partly by allowing solo players to " decorate " the melody they were playing—*solo improvisation*; or several players to indulge in their extemporization simultaneously—*collective improvisation*. The weaving of patterns by the three melody instruments may be compared with a graph in which the rhythm section lays down the coordinates, over which is traced the scarlet line of the cornet; the bright green of the clarinet, and the dark brown undulations of the trombone. A fanciful simile, perhaps, but one which explains the necessity for the beat in jazz to act as a scaffolding.

It is not strictly accurate, however, because the vertical and horizontal lines of the graph are themselves varied slightly in their placing, owing to rhythms within rhythms constructed by the different instruments of the rhythm section. It is as though four fine lines were drawn very closely together; when viewed at a distance of six feet they merge into one thickish line, but when inspected closely their separation is obvious.—Rex Harris, *Jazz* (Penguin Books, 1952), p. 53. [Per Dr. S. Rushton.]

THE SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS BY ITERATION.

By E. H. Bateman.

Abstract.

This note gives a general 2nd order solution, with the corresponding 3rd order correction, by means of which a good numerical answer can be obtained from an approximate estimate in one operation, and results of high accuracy by one or two iterations. The method is applied to integral roots of numbers, and to common types of equations.

For the extraction of square roots and cube roots of high accuracy, the general method is developed in the form of series designed to simplify the computation, and one example is given in full with an accuracy of over 30 significant figures.

1. *Theoretical development.*

If a is an approximate solution of the equation,

$$f(x)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

$x=a+\lambda$, where λ is an unknown error. If λ is small, the expansion of $f(a+\lambda)$ by Taylor's theorem gives the following equation for finding λ,

$$0=f(a)+\lambda f_1(a)+\lambda^2/2f_2(a)+\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

Neglecting the higher powers of λ,

$$\lambda \approx -\frac{f(a)}{f_1(a)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

which is Newton's approximation, originally derived from a geometrical system and now given in all the relevant text books, *e.g.* Burnside and Panton, and Whittaker and Robinson, (Refs. 1 and 2).

If one more term is included from the right-hand side of (1.2) there is a quadratic equation in λ, which is presumably the reason why no one has hitherto made use of it as the natural sequel to (1.3). But this quadratic is easily resolved for the purpose of approximate numerical solution by using (1.3) to reduce it to a linear form in λ. Thus, if

$$\phi=-\frac{f_0}{f_1-f_0f_2/2f_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

$$\lambda \approx \phi(a).$$

Now, if a and b are consecutive approximations to the solution of (1.1),

$$b=a+\phi(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

and if $e(a)$ and $e(b)$ are the respective errors in these approximate solutions,

$$x=a+e(a)=b+e(b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.6)$$

and substituting in (1.5)

$$e(b)=e(a)-\phi(a).$$

Next, substituting $\phi\{x-e(a)\}$ for $\phi(a)$ and expanding in powers of $e(a)$, we find after a little algebra

$$e(b)\approx e^3(a)\theta(x), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.7)$$

approximately, neglecting $e^4(a)$ and higher powers, where

$$\theta=\left(\frac{f_2}{2f_1}\right)^2-\frac{f_3}{6f_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Equation (1.7), showing that $e(b)$ depends primarily on the cube of $e(a)$,

emphasises the rapidity of convergence obtained by using equation (1.5) as an iterative relation, but sometimes it is worth while to combine (1.5) and (1.7) in order to obtain the improved approximation b', given by

$$b' = a + \phi(a) + \phi^3 \cdot \theta(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.9)$$

where $\phi(a)$ and $\theta(a)$ are approximations for $e(a)$ and $\theta(x)$.

In some of the examples which follow, it is shown that one application of (1.9) operating on a very rough approximation for (a) will give the numerical solution to an accuracy of 6 or 7 significant figures.

2. *Integral roots of numbers.*

If
$$f(x) = x^n - N,$$

$$\phi(x) = \frac{N - x^n}{nx^{n-1} + (n-1)(N - x^n)/2x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.11)$$

$$\theta(x) = \frac{n^2 - 1}{12x^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.12)$$

Then, for taking a fifth root, $n = 5$,

$$\phi(a) = \frac{R}{5a^4 + 2R/a}, \text{ where } R = N - a^n, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.21)$$

$$\theta(a) = 2/a^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.22)$$

As an example, take $N = 2$, and $a = 1$. Then $R = 1$,

$$\phi(a) = \tfrac{1}{7} = 0{\cdot}14286\,; \quad \phi^3 \cdot \theta(a) = \tfrac{2}{3\,4\,3} = 0{\cdot}00583,$$

giving 1·14869, which is within 1×10^{-5} of the true value.

For cube roots,

$$\phi(a) = \frac{R}{3a^3 + R/a}\,; \quad \theta(a) = \frac{2}{3a^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.3)$$

and again taking $N = 2$, and $a = 1$, $R = 1$, $\phi = \frac{1}{4}$, $\theta = \frac{2}{3}$,

$$b' = 1 + \tfrac{1}{4} + \tfrac{1}{4\,4} \times \tfrac{2}{3} = 1{\cdot}26.$$

This happens to be correct within 1 digit in the 4th place of decimals, and one further approximation gives the answer correctly to 17 significant figures; thus,

$$R = -0{\cdot}000\,376\,; \qquad \theta = \tfrac{2}{3} \times \frac{1}{1{\cdot}5876}\,;$$

$$\begin{aligned} b' &= \quad 1{\cdot}26 \\ \phi &= -0{\cdot}0000\ 7895\ 0104\ 9202 \\ \phi^3\theta &= -0.\qquad\qquad\qquad\quad 2066 \\ 2\tfrac{1}{3} &= \quad 1{\cdot}2599\ 2104\ 9894\ 8732 \end{aligned}$$

Here, the computation of ϕ involves a division by 12 figures which is a somewhat unreasonable sum, and for results of higher accuracy a more convenient method is given in section 3.

For square roots the iteration operators are

$$\phi(a) = \frac{R}{2a + R/2a}\,; \quad \theta(a) = \frac{2}{4a^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.4)$$

and taking 10 as a first estimate of $\sqrt{144}$,

$$\phi = \frac{44}{22{\cdot}2} = 1{\cdot}9820\,; \quad \phi^3\theta = \frac{1{\cdot}98^3}{400} = 0{\cdot}0194,$$

giving 12·0014, which is a good approximation considering the crudity of the initial estimate.

3. *Square roots and cube roots of higher accuracy.*

As already noted in evaluating the cube root of 2 to 17 significant figures, the use of a divisor of many figures does not promote rapid computation, with or without a machine. This defect, however, seems to be common to all the well-known methods of root taking, even including the version of extracting a square root recently put forward by Haldane (Ref. 3), who contemplates accuracies of 50 or 100 figures.

In order to avoid the long divisor, we note that $\phi(a)$ is a function of R/a, so that it can be expressed as a series of power of R/a. This will still leave powers of (a) in the denominator and we therefore expand in powers of R/N. Thus, for square roots,

$$N^{\frac{1}{2}} = a + \frac{R}{2a} - \frac{R^2}{8aN} - \frac{R^3}{16aN^2} - \frac{5R^4}{128aN^3} - \frac{7R^5}{256aN^4}. \quad \ldots\ldots\ldots\ldots(3.1)$$

An alternative series in which (a) is completely eliminated from the denominator is

$$N^{\frac{1}{2}} = a\left(1 + \frac{r}{2} + \frac{3r^2}{8} + \frac{5r^3}{16} + \frac{35r^4}{128} + \frac{63r^5}{256}\right) + e, \quad \ldots\ldots\ldots\ldots(3.2)$$

where $r = 1 - a^2/N$, e is approximately $\frac{5}{16}ar^6$.

For cube roots the corresponding series is

$$N^{\frac{1}{3}} = b\left(1 + \frac{r}{3} + \frac{2r^2}{9} + \frac{14r^3}{81} + \frac{35r^4}{243} + \frac{273r^5}{2187}\right) + e, \quad \ldots\ldots\ldots\ldots(3.3)$$

where $r = 1 - \dfrac{b^3}{N}$, e is approximately $\frac{1}{9}br^6$.

As an example of the use of these series, consider the cube root of 2 to 24 places taking a value of b correct to nearly 4 decimal places :

$N = 2$, $b = 1{\cdot}2600$, $R = 2 - (1{\cdot}26)^3 = -0{\cdot}000376$; $r = -0{\cdot}000188$.

				8	12	16	20	24	
$r = -(3)$	1,88	1/3	−	6266	6666	6666	6666	6666	67
$r^2 = +(7)$	3,5344	2/9	+		7854	2222	2222	2222	22
$r^3 = -(11)$	664,4672	14/81	−		1	1484	6182	7160	49
$r^4 = +(14)$	12,4919,8336	35/243	+			1	7992	5686	26
$r^5 = -(18)$	23,4849,29...	273/2187	−				2	9315	89
$r^6 = +(22)$	44,15...	1/9	+					4	91
			−	6266	6667	8151	2852	3143	05
			+		7854	2224	0214	7913	39
			−	6265	8813	5927	2637	5229	66
		× 1·26	−	7895	0105	1268	3523	2789	37
$\sqrt[3]{2} =$		1·2599		2104	9894	8731	6476	7210	63

which agree with the result, rounded to 7211 at 24 places, calculated by Peter Gray in 1876 (Ref. 4).

Next consider the square root of 1·37, the number chosen by Haldane to exemplify his method.

First pick a 3 figure answer, by any method, *i.e.* 1·17. Then using the iteration formula (1.9),

$$b = a + \phi(a) + \phi^2 \cdot \theta(a)$$
$$a^2 = (1{\cdot}17)^2 = 1{\cdot}3689\ ; \quad R = 0{\cdot}0011$$
$$b = 1{\cdot}17 + \frac{0{\cdot}0011}{2{\cdot}3404\ 7009} + \frac{(9)103817}{5{\cdot}48}$$
$$= 1{\cdot}1704\ 6999\ 1071.$$

The second estimate is therefore taken as 1·17047 with an error of $0{\cdot}893 \times 10^{-8}$.

Since
$$r < \frac{2 \times 0{\cdot}893}{1{\cdot}171} \times 10^{-8} < 1{\cdot}55 \times 10^{-8},$$
$$r^6 < 14 \times 10^{-48},$$
$$e < 4 \times 10^{-48},$$

and if e is estimated to 3 significant figures the series in equation (3.2) will give the answer to 50 places of decimals.

As an illustration of the simplicity of the method, the computation of the first 30 places is given in full detail :

$$a = 1{\cdot}17047; \quad a^2 = 1{\cdot}3700\ 0002\ 09\ ; \quad R = -(7)209$$
$$\frac{R}{2} = \quad -(7)1\ 0450$$
$$-\frac{R^2}{8N} = \quad -(16)\ 3985\ 4927\ 0072\ 9927$$
$$-\frac{R^3}{16N^3} = -\frac{R}{2} \cdot \frac{R^2}{8N} \cdot \frac{1}{N} = \quad +(24)\ 3040\ 0291$$

$\Sigma =$	$-(7)1$	0450	0000	3985	4926	7032	9636
$\Sigma/1{\cdot}17047 =$	$-(8)$	8928	0374	8908	9790	1725	7713

$\sqrt{1{\cdot}37} = 1{\cdot}1704$ 6999 1071 9625 1091 0209 8274 2287

In this particular example of taking the square root to a moderate number of places, the series (3.1) gives a neater solution because there is no great difficulty in dividing by a six-figure number, but the series (3.2) is definitely better if a large number of places is required.

4. *The solution of algebraic equations.*

First of all considering the " celebrated equation of Wallis " so called because " it was the one on which Wallis chanced to exhibit Newton's method when he first published it, in consequence of which every numerical solver has felt bound in duty to make it one of his examples " (Ref. 2),

$$f(x) = x^3 - 2x - 5 = 0,$$

which has a root near 2. Let $a = 2$.

$$f_1(x) = 3x^2 - 2\ ; \quad f_2(x) = 6x\ ; \quad f_3(x) = 6.$$
$$f_1(a) = 10\ ; \quad f_2(a) = 12\ ; \quad f_3(a) = 6.$$
$$f_0(a) = -1\ ; \quad \phi = \frac{1}{10{\cdot}6}\ ; \quad a + \phi = 2{\cdot}094340.$$
$$\theta = \frac{144}{400} - \frac{6}{60} = 0{\cdot}26\ ; \quad \begin{matrix} \phi^3\theta = 0{\cdot}000218 \\ b = 2{\cdot}094558 \end{matrix}.$$

This first approximation is within 0·000007 of the correct answer with a proportionate error of less than $3{\cdot}5 \times 10^{-6} = e$, and since the error in the second

approximation is approximately e^4, this second approximation will be correct to 20 places of decimals, showing a far higher rate of convergence than any of the methods described in the text books.

To show that this is no accidental result, consider

$$f(x) = x^3 - 4x^2 + 5 = 0$$

which Whittaker and Robinson use to exemplify Horner's method. For the root between 1 and 2, since $f_0(1) = 2$ and $f_0(2) = -3$, $a = 1{\cdot}4$ is the obvious first estimate. Then

$$f_0(a) = -0{\cdot}096\,; \quad f_1(a) = -5{\cdot}32\,; \quad f_2(a) = 0{\cdot}4\,; \quad f_3(a) = 6.$$

$$a + \phi = 1{\cdot}4 + \frac{0{\cdot}096}{-5{\cdot}32 - \dfrac{0{\cdot}096}{5{\cdot}32} \times \dfrac{0{\cdot}4}{2}} = 1{\cdot}4 - 0{\cdot}0180\ 329$$

$$= +1{\cdot}3819\ \ 671$$

$$\theta = \frac{0{\cdot}16}{10{\cdot}64^2} + \frac{1}{5{\cdot}32} = 0{\cdot}189\,; \qquad \begin{aligned} \phi^3\theta &= -0{\cdot}0000\ 011 \\ b &= \ \ 1{\cdot}3819\ 660 \end{aligned}$$

with an error of 2×10^{-7}, which is the same order of accuracy as that obtained by Whittaker and Robinson after comparatively heavy computing.

5. *Solution of a transcendental equation.*

The method of section 1 can be applied equally well to transcendental equations. Thus, for example, suppose we require the lowest root of the frequency equation

$$f(x) = 1 + \cos x \cosh x = 0$$

if values of $(\cos x \cosh x)$ and its first three derivatives are tabulated at intervals of 0·2 of the argument (Ref. 5).

From inspection of the table, the nearest entry is 1·8 where the four relevant values are, for f_0, f_1, f_2, f_3,

$$0{\cdot}2940\,; \quad -3{\cdot}6947\,; \quad -5{\cdot}7304\,; \quad -4{\cdot}7156.$$

$$a = 1{\cdot}8\,; \quad \phi(a) = -\frac{0{\cdot}2940}{-3{\cdot}6947 - \dfrac{0{\cdot}147 \times 5{\cdot}7304}{3{\cdot}6947}} = \frac{0{\cdot}2940}{3{\cdot}9227}.$$

$$\theta(a) = \left(\frac{5{\cdot}73}{7{\cdot}39}\right)^2 - \frac{4{\cdot}72}{22{\cdot}2} = 0{\cdot}60 - 0{\cdot}21 = 0{\cdot}39.$$

$$b = a + \phi + \phi^3\theta = 1{\cdot}8 + 0{\cdot}07495 + 0{\cdot}00016$$
$$= 1{\cdot}8751$$

which is correct to this degree of accuracy.

6. *Geometrical interpretation.*

Geometrically, the problem is to find the abscissa OS, where S is the point at which the curve PQS intersects the axis of x.

If $y = f(x)$ is the equation of the curve,
$AP = f(a)$, where $a = OA$.

Then
$$\frac{AP}{BA} = f_1(a),$$

and
$$OB = OA - BA = a - \frac{f(a)}{f_1(a)}$$

which is of course Newton's approximation.

If now N is the mid-point of BA and QN the corresponding ordinate, the change of slope between P and Q is

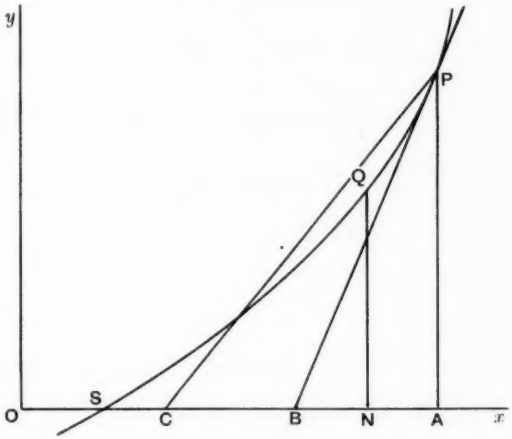
$$f_2(x)\,\delta x = f_2(a)\,\frac{AB}{2} = f_0 f_2/2f_1$$

and the slope at Q is $f_1 - f_0 f_2/2f_1$.

Now draw PC the line through P parallel to the tangent at Q cutting the x axis in C,

$$\begin{aligned} OC &= OA - CA \\ &= OA - \frac{AP}{\tan PCA} \\ &= a - \frac{f(a)}{f_1 - f_0 f_2/2f_1}. \end{aligned}$$

Hence OC is the solution given by equation (1.5), and the corrected solution of equation (1.9) gives a point nearly coincident with S on the scale of the diagram.



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E. H. B.

ROD, POLE AND PERCH.

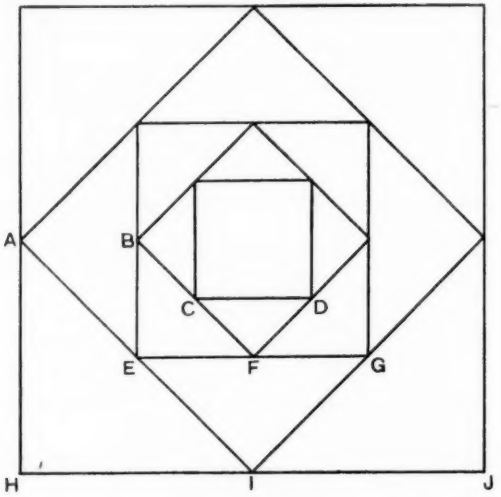
By "Peter Simple".

Nihil enim potest esse aequabile, quod non a certa ratione proficiscatur.
Cicero.

Filius : Daddy! In Arithmetic to-day Old Syphon taught us that $5\frac{1}{2}$ yards make one rod, pole or perch. Isn't it silly? *Three* names for a ridiculous length like $5\frac{1}{2}$ yards!

Pater : I do wish you would not say *Arithmetic* when you allude to Mensuration. . . . But did not Doctor Cypher explain that the pole is one-quarter of a cricket-pitch ; which pitch is set out with a length of chain one-tenth of a furlong long?

Filius : No! A quarter of a cricket-pitch seems pretty decent to me. Was the pole really made that way?



Pater : No ; the chain and the furlong were adjusted* to suit the pole. The prime measure was the inch—three barley-corns from the middle of the ear—now one-thirty-sixth of the imperial yard.

Filius : But a pole is 198 inches. Why not a nice round number like 200? How potty† our ancestors must have been! Did they run out of barley-corns?

Pater : Not at all! 198 is a *preferred* number ; its square is very very nearly twice the square of 140, and very very nearly half the square of 280.

* *wangled.* † *inconsequential.*

FILIUS : Is it really?

$$198^2 = 39204 = 2 \times (19602) = 2 \times (140{\cdot}007)^2$$

Pythagoras! 90° 0′ 20″! What a nifty approximation!

PATER : You will now understand all about rods, poles and perches. Regard the figure :

$$EG = 140 \text{ inches} = 11\tfrac{2}{3} \text{ ft.} = \textit{rod}$$
$$AI = 198 \text{ inches} = 16\tfrac{1}{2} \text{ ft.} = \textit{pole}$$
$$HJ = 280 \text{ inches} = 23\tfrac{1}{3} \text{ ft.} = \textit{perch}$$

The rod is derived from the Pharaonic *qanu* of $11\frac{2}{3}$ Egyptian feet. Whence came also the measuring *reed*, described by Ezekiel, of 6 great-cubits. The great-cubit, a cubit and an hand-breadth, was 7/6 of a cubit, which cubit measured 5/3 of a foot ; the reed, therefore, was $11\frac{2}{3}$ Israelitic feet.

The perch is sometimes called a *rood* and the many synonyms for the pole include *lug*, *virgate* and *yard*. We can discriminate the perch from the pole as it is extant in the French *perche*, 22 *pieds-du-roi*, which contains 281·4 of our inches.

CD, half the rod, is the 70-inch *staff** (which was probably the measure of the old *fathom*) and we may surmise that *BF*, half the pole, is the *quarterstaff*.†

FILIUS : I say! This is jolly good! I wanted to know about quarterstaffs. But you haven't said why I have to learn " $5\frac{1}{2}$ yards make one rod, pole *or* perch "?

PATER : Oh! that is probably due to the slovenly drafting of some bygone by-law. Can't you imagine it? " Every house shall be properly set out with a rod, pole or perch ", instead of " . . . with a rod, a pole or a perch ". That is how things happen. Now run along and get on with your nuclear physics.

FILIUS : O.K., Daddy!

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

* *eight-foot-and-a-quarter* staff.

† The staff is a *sexpeda*—a measuring-rod *vi pedes* long. Inasmuch as

$$xx \textit{ sexpedae} \equiv xii \textit{ decempedae} \equiv \textit{actus} = 5 \text{ perches} \equiv 10 \text{ rods} \equiv 20 \text{ staves},$$

the induction is ineluctable that the inch derived from the *sexpeda* and was an aliquot thereof.

THE CALCULATION OF LARGE PRIMES.

By E. M. Wright.

For the last 75 years the largest known prime has been the Mersenne number $2^{127}-1$, proved prime by Lucas. In two recent notes (*Eureka*, October 1951 and *Nature*, 168 (1951), 838), Miller and Wheeler announced their identification of several larger primes, of which the greatest is

$$180(2^{127}-1)^2+1. \qquad (1)$$

Their method depends upon a special form of converse of Fermat's Theorem. This provides a test for the primality of numbers of the form $n=kp+1$ and $n=kp^2+1$. The most laborious part of the test is the determination whether or not

$$2^{n-1}\equiv 1 \pmod{n} \qquad (2)$$

in any particular case; this was carried out on Edsac at Cambridge.* Here I show that we can determine the primality of kp^2+1 in the same way. The theory is still very elementary.

In what follows all latin letters (small or large) are integers (usually positive) and $p, P, p_1, p_2, \ldots$ are odd primes. We write $a \mid b$ to denote that a divides b, *i.e.* $b=ac$ for some c, and $a \nmid b$ for the negative of this. Again $\phi(n)$ is Euler's function, so that, if

$$n=p_1^{a_1}p_2^{a_2}\ldots p_r^{a_r} \qquad (3)$$

is the usual expression of n as a product of powers of different primes, we have

$$\phi(n)=p_1^{a_1-1}\ldots p_r^{a_r-1}(p_1-1)\ldots(p_r-1). \qquad (4)$$

Lemma 1. *If a is prime to n and if d is the least positive value of x for which*

$$a^x\equiv 1 \pmod{n}, \qquad (5)$$

then (5) *is true for a particular x if, and only if, $d \mid x$. Also $d \mid \phi(n)$.*

This is well-known; see, for example, Theorem 88 of Hardy and Wright, *Introduction to the Theory of Numbers* (second edition, Oxford 1945).

Lemma 2. *If $b \geqslant 1$, $n=kp^b+1$ and*

$$2^k \not\equiv 1, \quad 2^{n-1}\equiv 1 \pmod{n}, \qquad (6)$$

then n has a prime factor $P \equiv 1 \pmod{p}$.

By (6) n is odd and we may put $a=2$ in Lemma 1. Hence $d \nmid k$, $d \mid (n-1)$, *i.e.* $d \mid kp^b$ and so $p \mid d$. Again, $d \mid \phi(n)$ and so $p \mid \phi(n)$. But $p \nmid n$ and so, if n is expressed in the form (3), p divides one of $p_1-1, p_2-1, \ldots, p_t-1$, say $P-1$. This is the lemma.

If n is composite, we have $n=Pm$, $m>1$, $m=n/P\equiv 1 \pmod{p}$.

Hence
$$n=(up+1)(vp+1), \quad 1\leqslant u\leqslant v, \qquad (7)$$

that is,
$$kp^{b-1}=uvp+u+v, \quad 1\leqslant u\leqslant v. \qquad (8)$$

Lemma 3. *If $b=1$ or 2 and $k<p$, then* (8) *is impossible.*

For, suppose (8) true and $k<p$. If $b=1$, by (8) $p\leqslant uvp<kp^{b-1}=k<p$, which is impossible. If $b=2$, then by (8),

$$kp=uvp+u+v, \quad p \mid (u+v),$$

and so
$$2v\geqslant u+v\geqslant p, \quad uv\leqslant k-1\leqslant p-2,$$

* Recently the applicatian of Lucas's test for primes of the form 2^p-1 (by Prof. R. M. Robinson using SWAC) has led to the discovery of several very much larger primes.

$$1 \leqslant u \leqslant \frac{p-2}{v} \leqslant \frac{2(p-2)}{p} < 2, \quad u = 1,$$

$$v \geqslant p - u = p - 1, \quad v \leqslant \frac{p-2}{u} = p - 2,$$

a contradiction.

From Lemmas 2 and 3, we have :

Theorem 1. *If $k < p$, $n = kp + 1$ or $kp^2 + 1$, and* (6) *is true, then n is prime.*

It was by this theorem that Miller and Wheeler found that (1), a number of 79 digits, was prime.

We now take $b = 3$, so that $n = kp^3 + 1$. If k is odd, n is even and so composite. Again, if $k = w^3$, we have the trivial factorisation

$$kp^3 + 1 = (wp + 1)\{(w^2p - w)p + 1\}$$

of the form (7). We therefore suppose that k is even, not a perfect cube and less than p. If (8) is true,

$$kp^2 = uvp + u + v, \quad u + v = gp, \quad g \geqslant 1,$$

$$kp = uv + g = ugp - u^2 + g, \qquad u^2 \equiv g \pmod{p},$$

$$u^2 \leqslant uv < kp < p^2, \quad 1 \leqslant u < p,$$

$$kp > u(gp - u) > (g-1)p, \quad g < k + 1 \leqslant p.$$

Hence
$$u^2 \equiv g \pmod{p}, \quad 1 \leqslant u < p, \quad 1 \leqslant g < p \qquad \text{.....(9)}$$

and so
$$u^2 = jp + g, \quad j \geqslant 0, \quad k = ug - j. \qquad \text{.....(10)}$$

Hence, to modulus 2,
$$0 \equiv k \equiv ug + j \equiv u^2g + j \equiv gjp + g^2 + j$$
$$\equiv gj + g + j \equiv (g+1)(j+1) - 1.$$

Thus g and j are even and so, by (10), u also. Again, if $j = 0$, $u^2 = g$ and $k = ug = u^3$, a cube, which is impossible. Hence $g \geqslant 2$, $j \geqslant 2$. Also, by (10),

$$j = \frac{u^2 - g}{p} < \frac{u^2}{p} < u \qquad \text{.....(11)}$$

and so

$$2 \leqslant g < p, \quad 2 \leqslant j < u < p. \qquad \text{.....(12)}$$

Now, let us write $U = \sqrt{2p + g}$, so that

$$U^2 = 2p + g \leqslant jp + g = u^2 < p^2, \quad 0 < U < u < p$$

and

$$(j-2)(u+U) < 2p(j-2) = 2(u^2 - U^2),$$

so that

$$j - 2 < 2(u - U) \leqslant g(u - U),$$

and

$$k = gu - j > gU - 2 \geqslant 2\sqrt{2p+2} - 2. \qquad \text{.....(13)}$$

Hence, we have :

Lemma 4. *If $b = 3$ and if k is even, not a cube and less than $2\sqrt{2p+2} - 2$,* then* (8) *is impossible.*

From this and Lemma 2, we deduce at once

Theorem 2. *If $n = kp^3 + 1$, if* (6) *is true and if k is even, not a cube and less than $2\sqrt{2p+2} - 2$, then n is prime.*

* This implies that $k < p$ since $2\sqrt{2p+2} - 2 < p$ for $p > 3$, while, for $p = 3$,

$$2\sqrt{2p+2} - 2 < 4$$

and so $k = 2 < 3$.

The condition $k < 2\sqrt{2p+2} - 2$ is necessary, if (8) is to be impossible. If we put $p = 7$ and $k = 6$, we have

$$k = 6 = 2\sqrt{2p+2} - 2$$

and
$$6 \cdot 7^3 + 1 = (4 \cdot 7 + 1)(10 \cdot 7 + 1). \quad \ldots\ldots(14)$$

Indeed, the same is true with $p = 2^{127} - 1$ and

$$k = 2\sqrt{2p+2} - 2 = 2^{65} - 2,$$

when
$$kp^3 + 1 = (2^{64}p + 1)\{(2p - 2^{64})p + 1\}. \quad \ldots\ldots(15)$$

Now let us suppose that (8) is true, that k is even and not a cube and that

$$k < \min\,(p, 6\sqrt{2p+6} - 2). \quad \ldots\ldots(16)$$

By (13) and (16), we have

$$6\sqrt{2p+6} > gU = g\sqrt{2p+g},$$

so that $g < 6$. Hence $g = 2$ or 4. If $g = 4$, by (9), $u^2 \equiv 4 \pmod{p}$, $1 \leqslant u < p$, whence $u = 2$ or $p - 2$. The latter is odd and the former makes $j = 0$; both are therefore impossible. Hence $g = 2$.

We have now $u^2 = jp + 2$, $k = 2u - j > u$, since $j < u$ by (11). Hence, by (16),

$$u < k < 6\sqrt{2p+6} - 2 < 6\sqrt{2p+6},$$
$$jp + 2 = u^2 < 72p + 216,$$
$$j(j - 72) < p(j - 72) < 214,$$

and so
$$j < 75.$$

Since $u^2 = jp + 2$, 2 must be a quadratic residue mod j and mod p. Hence, $4 \nmid j$, and p and every odd prime factor of j is of the form $8m \pm 1$. (See, for example, Hardy and Wright, *loc. cit.*, Theorem 95). The only possible j are therefore $j = 2, 14, 34, 46, 62$.

We have from (16),

$$k = 2\sqrt{jp+2} - j < 6\sqrt{2p+6} - 2 \quad \ldots\ldots(17)$$

and this gives us an upper bound for p when $j = 34$, 46 or 62. Of all the possible p for these j, only $p = 47$, $j = 34$ makes $jp + 2 = 1600$ a square. Hence we have just the one factorisation

$$46 \cdot 47^3 + 1 = (40 \cdot 47 + 1)(54 \cdot 47 + 1).$$

For $j = 2$ or 14, (17) provides no upper bound for p. If $j = 2$ and $u = 2w$, we have

$$p = 2w^2 - 1. \quad \ldots\ldots(18)$$

For every p of this form, there is one even k, *viz.* $k = 4w - 2$, which satisfies (17) and leads to a factorisation (7), viz.

$$(4w - 2)p^3 + 1 = (2wp + 1)\{2(p - w)p + 1\}.$$

(14) and (15) are examples of this.

If $j = 14$, we see that for every $p \neq 7$ of the form

$$p = (2w^2 - 1)/7, \quad \ldots\ldots(19)$$

there is one even k, viz. $k = 4w - 14$, which satisfies (17) and leads to a factorisation (7), viz.

$$(4w - 14)p^3 + 1 = (2wp + 1)\{2(p - w)p + 1\}.$$

We remark that no p (except 7, too small for $j = 14$) is of both the forms (18) and (19). It is unknown whether there is an infinity of primes of the form (18) or of the form (19), but both are highly probable. E. M. W.

ON COMMUTATIVE MATRICES.

By M. F. Egan. & R. E. Ingram

In the *Journal of the London Mathematical Society* (July, 1951, Vol. 26, pp. 221-8), proofs are given of a theorem of McCoy's and of various results that follow from it. McCoy's theorem establishes the necessary and sufficient condition that a number of $n \times n$ matrices should be reducible to triangular form by the same transformation $T^{-1}(\ )T$, where the matrix T is unitary. (We define a triangular matrix to be one in which all the elements below the diagonal are zero.)

The methods of proof used in the *Journal* are more elementary than those originally employed. Still, to one reader at least the article was very heavy going, and in what follows we give simpler proofs of some of the results, though not of the main theorem.

Let x and y be two vectors, and let $x = Tu, y = Tv$. Then if $Ax = y$, $ATu = Tv$, and $T^{-1}ATu = v$. In particular, if $y = \lambda x$, $v = \lambda u$, so that the eigenvectors for a given latent root correspond. Again, the rth column-vector of T, say t_r, is Te_r, where e_r is the rth basic unit vector ; so that if t_r is an eigenvector of A for the root λ, e_r is an eigenvector of the transform of A (which we may call tA) for the same root, and the rth column of tA will have λ in the rth row and zero in all the others. So, if A has a complete set of n independent eigenvectors, and we take these as the column-vectors of T, tA will be diagonal.

In general, any $n \times n$ matrix A can be reduced to triangular form by a matrix T, (which we may if we choose require to be unitary). Take as the first column-vector of T an eigenvector of A for the root λ ; we shall have

$$tA = \begin{pmatrix} \lambda & z' \\ 0 & B \end{pmatrix},$$

where z' is an $(n-1)$-row vector, and B is a square matrix of order $n-1$. If the statement is true for $n-1$, there is a matrix V which triangulates B. Hence, if

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix},$$

then $T_2 = TT_1$ triangulates A. But the statement is true for 2, since in that case z' and B are scalars ; hence it is true universally. We may if we choose require that the transforming matrix be unitary. We can, in an infinity of ways, construct a unitary matrix having as its first column a given vector x, normalized so that $\bar{x}'x = 1$; hence we can make T and V (and therefore also T_1 and T_2) unitary.

If T is unitary and $tA = A_1$, then $t\bar{A} = A_1'$. (Since $\overline{T}'AT = A_1$, $\bar{A}_1' = \overline{T}'\bar{A}'T$.) Also, if x and y are any two vectors, and $x = Tu$, $y = Tv$, then $\bar{x}'y = \bar{u}'\overline{T}'Tv = \bar{u}'v$, so that the scalar product is unchanged. If this product is zero, the vectors are said to be perpendicular. Again, if two matrices commute, it is easily seen that their transforms by any matrix T, unitary or not, commute also. It follows that if a matrix A is normal, *i.e*, it commutes with $\bar{A}'$, its transform by a unitary matrix is also normal. Hence we deduce :

I. *The necessary and sufficient condition that a matrix should be transformable to diagonal form by a unitary matrix is that it should be normal.*

The condition is clearly necessary, since a diagonal matrix is normal, and this property is invariant for a unitary transformation. To prove it sufficient : we can reduce our matrix to triangular form by a unitary transformation ; and a triangular normal matrix is diagonal. To prove this, let A be such a matrix. Then

$$A = \begin{pmatrix} \lambda & z' \\ 0 & B \end{pmatrix}, \quad A' = \begin{pmatrix} \bar{\lambda} & 0 \\ \bar{z} & B' \end{pmatrix}.$$

Since A commutes with A', we find that (i) $z'\bar{z} = 0$, whence $z = 0$; and (ii) B is normal. Since B is triangular, then if the proposition is true for $n-1$, B is diagonal, and, since $z = 0$, A is also diagonal. Hence, etc.

II. *If the matrices $A_1, A_2, A_3, \ldots$ commute, they can be reduced to triangular form by a common T-transformation*, and we may if we choose require T to be unitary.

We first prove that they have a common eigenvector. For this we need a lemma, namely : If a matrix transforms among themselves the vectors of the sub-space determined by p independent vectors, it has an eigenvector in this sub-space.

It is easy to construct a formal proof of this, but the thing is really evident. For the existence of eigenvectors of a matrix M in the n-space results from the fact that M is an operator which transforms the vectors of the space linearly among themselves. (We shall have occasion to give a more precise statement of this point later, under IV.)

Let λ_1 be a latent root of A_1. If it is a multiple root, there are, say p independent eigenvectors for it. These determine a p-space, and the expressions " eigenvector of A_1 for λ_1 " and " vector of this p-space " are identical. Now if x is any vector of this space, then $A_1A_2x = A_2A_1x = \lambda_1A_2x$, so that A_2x is another vector of the p-space, and hence by our lemma A_2 has an eigenvector in this space, say for the root λ_2. If it has more than one for the same root, then A_1 and A_2 have a certain number of common eigenvectors for the two given roots. These again determine a subspace, in which A_3 has one or more eigenvectors for some root λ_3. The process can clearly be continued indefinitely.

Taking the common eigenvector as the first column of T, we have

$$tA_i = \begin{pmatrix} \lambda_i & z_i' \\ 0 & B \end{pmatrix},$$

and it is easily seen that the $(n-1) \times (n-1)$ matrices B_i commute. Hence, if the proposition is true for $n-1$, it is true for n. But it is true (and trivial) for 1. Hence, etc.

Corollary. If a number of normal matrices commute, they can be reduced to diagonal form by a common unitary transformation. For if a normal matrix is triangular, it is diagonal. In fact, the necessary and sufficient conditions that a number of matrices can be reduced to diagonal form by a unitary transformation are that they should be normal and should commute. For diagonal matrices satisfy both.

III. Given a number of matrices $A_1, A_2, A_3, \ldots$, and writing

$$C_{ij} = A_iA_j - A_jA_i,$$

then : *If A_i commutes with C_{jk} for all i, j, k concerned, the matrices can be reduced to triangular form by a common T-transformation*, which can be taken to be unitary if desired.

Lemma. If A commutes with $AB - BA = C$, all the latent roots of C are zero (and C is therefore nilpotent, since it satisfies its characteristic equation $\lambda^n = 0$).

Assuming this for the moment, we note that since A_j and A_k both commute with C_{pq}, so does C_{jk}. And since A_i and all the C_{jk} commute, they have at least one common vector for which

$$A_ix = \lambda_ix, \quad C_{jk}x = 0.$$

Now, if it is true that we have one or more vectors x for which $A_i x = \lambda_i x$, $(i = 1, 2, \ldots, p-1)$; $C_{jk}x = 0$ for all j, k; then A_p transforms among themselves the vectors of the space defined by these x; for we have

$$A_i A_p x = A_p A_i x = \lambda_i A_p x \quad \text{and} \quad C_{jk} A_p x = A_p C_{jk} x = 0.$$

Hence there is one of the vectors x for which $A_p x = \lambda_p x$, and the proposition holds for p matrices if it holds for $p-1$ (namely that they have a common eigenvector with the C_{jk}). But it holds for one, therefore universally.

Taking as before this vector as the first column-vector of T, we get the same expression for tA_i as we did under II, and it is not hard to see that in this case also the B_i have the property of the A_i; hence the proof proceeds as before.

The following very neat proof of the lemma is given in the *Journal* article as due to Mr. J. E. Bullard: The sum of the latent roots of a matrix is equal to its "trace", (the sum of its diagonal elements). It is easily seen that AB and BA have the same trace, hence the trace of C is zero. Again, if A commutes with C, we have

$$C^{p+1} = C^p AB - C^p BA = A(C^p B) - (C^p B)A,$$

and hence has trace zero. Accordingly, if C has k distinct non-zero roots $\lambda, \mu, \ldots, \omega$ with multiplicities $m_1, m_2, \ldots, m_k$, we have

$$m_1\lambda^p + m_2\mu^p + \ldots + m_k\omega^p = 0, \quad (p = 1, 2, 3, \ldots).$$

Taking the first k of these equations to determine $m_1, m_2, \ldots, m_k$, the determinant of these equations does not vanish, since λ, μ, ν, etc. are all supposed different from zero and from one another; hence we get $m_1 = m_2 = \ldots = m_k = 0$, and there are no non-zero roots.

IV. *The necessary and sufficient conditions that a number of matrices should be reducible to diagonal form by a common T-transformation are* (i) *that they should each have a complete set of n independent eigenvectors, and* (ii) *that they commute.*

The two conditions are clearly necessary, since diagonal matrices satisfy them. To prove them sufficient it will be enough to show that the matrices have a complete set of eigenvectors in common. To this end, we note that if a matrix M transforms among themselves the vectors v_1 of the sub-space determined by the first n_1 basic vectors, the first n_1 columns of M consist of a square matrix M_1 followed by zeros. For the rth component of Mv_1 is got by multiplying the n_1 components of v_1 by the first n_1 elements of the rth row of M; and by our supposition this product will be zero for every v_1 if $r > n_1$; whence the result. Again, if M transforms among themselves the vectors v_2 determined by the next n_2 basic vectors, it is easily seen that the next n_2 columns of M consist of the square matrix M_2 immediately following M_1 on the diagonal line, with zeros above and below. Accordingly, if the basic vectors are divided into successive sets of $n_1, n_2, \ldots, n_k$ vectors, and M transforms among themselves the vectors v_r determined by the rth set $(r = 1, 2, \ldots)$, then we can write

$$M = \text{Diag}\,(M_1, M_2, \ldots, M_k),$$

or, let us say, Diag (M_r).

Again, since $Mv_r = M_r v_r$, every eigenvector of M_r is an eigenvector of M. Further, every eigenvector of M is the sum of a number of these. We can write $x = v_1 + v_2 + \ldots + v_k$, so that if $Mx = \mu x$, then

$$(M_1 v_1 - \mu v_1) + (M_2 v_2 - \mu v_2) + \ldots = 0,$$

and since the expressions in brackets are independent, each of them must vanish. It follows that the maximum number of independent eigenvectors

of M is the sum of the maxima numbers for the M_r. In particular, if M has n independent eigenvectors, every M_r has n_r.

Now, suppose that the matrices $A, B, C, \ldots$ satisfy our two conditions. If A has k distinct latent roots $\lambda_1, \lambda_2, \ldots$, with n_r eigenvectors for the root λ_r, we may suppose the matrices to have been subjected to a T-transformation, with the n eigenvectors of A taken in order as the column-vectors of T. A will then be diagonal, and its eigenvectors for the root λ_r are the vectors v_r of the rth sub-space. Again, $ABv_r = BAv_r = \lambda_r Bv_r$, so that Bv_r is an eigenvector of A for the root λ_r, and hence belongs also to the rth sub-space. It follows as above that $B = \text{Diag}\,(B_r)$, and so also for C, D, etc.

Since B has n independent eigenvectors it follows (as in the case of M above) that B_r has n_r, which are eigenvectors of B, and also of A for the root λ_r. Taking $r = 1, 2, 3, \ldots$, we see that A and B have a common set of n independent eigenvectors; and the proposition is true for two matrices.

Again, $BC = CB$, *i.e.* $\text{Diag}\,(B_rC_r) = \text{Diag}\,(C_rB_r)$, hence B_r and C_r commute; and the same holds for $D_r, E_r, \ldots$.

Now, suppose that if $m-1$ matrices commute and have each a complete set of independent eigenvectors, they have a common set; and consider m matrices $A, B, C, \ldots, M$, satisfying the conditions. We have seen that the $m-1$ matrices $B_r, C_r, \ldots, M_r$ commute and have each a complete set of n_r independent eigenvectors; they have therefore a common set, and these are all eigenvectors of A for the root λ_r. Taking $r = 1, 2, \ldots, k$, we find a common set for the m matrices $A, B, \ldots, M$. But we have seen that the proposition is true for two matrices, hence it is true universally.

To reduce the matrices to diagonal form, we have only to take the common set of eigenvectors as the columns of T.

M. F. E. & R. E. I.

CORRESPONDENCE.

AN ENIGMA

To the Editor of the *Mathematical Gazette*.

SIR,—Several correspondents have kindly explained the capital letters in the statement (Note 2333)

"$2x^2 - x - 1 = (2x+1)(x-1)$ by F.M.O.L.".

They form a mnemonic for multiplying together two binomials and stand for

Firsts, Middles, Outers, Lasts.

Another version is F.O.I.L., where I = Inners. I fear the suggestion that the candidate meant to write "by F.M.O.F.", meaning "by fair means or foul", must be rejected.

Yours, etc., C. O. TUCKEY.

A TRIANGLE FORMULA

To the Editor of the *Mathematical Gazette*.

SIR,—It is not often that the *Gazette* takes one back in memory to schooldays of more than seventy years ago.

The proof of the formula for $\tan \frac{1}{2}(B-C)$ on p. 50 of the current number (February, 1953) can be found on pp. 75–6 of *Trigonometry for beginners* by I. Todhunter, published in 1871; but it was not included in his larger work, *Trigonometry for the Use of Colleges and Schools*.

Yours, etc., A. S. RAMSEY.

THE APPROACH TO ALGEBRA.

By E. M. Renwick.

Beginning a new study involves considerations both of strategy and of tactics. The strategic objective is to impart to the study a sense of purpose, to arouse interest in it, to encourage the pupil to accept the challenge which is implicit in the new situation which confronts him. The tactical question involves the choice of method.

So far as strategy is concerned, the older student beginning algebra has no difficulty in recognising its practical importance in science and technology, where the power of its language to exhibit and clarify relational patterns is manifested. The appeal by way of vocational and cultural interests has been made successfully in various popular books for adults and "second-tryers"; these interests simplify the teaching problem, and the teacher can proceed at an early stage to give training in analysis and expression.

With young beginners the problem is more complex. This is clear from the various Reports of the Mathematical Association, in which it is possible to detect a change of emphasis between 1934, when the *Report on the Teaching of Algebra in Schools* appeared, and 1948/9, years which saw the publication of the Reports on *Preparatory School Mathematics*, and on the *Teaching of Mathematics in Secondary Modern Schools* (an interim report). In the older publication, consideration is given chiefly to the outlook which it is desirable to encourage in the pupil towards those later developments of the subject which successful adaptation to the teaching will bring into being; in the more recent issues we find that the psychological effect of failure or success in the early stages is regarded as supremely important. Three quotations will illustrate this difference in emphasis: (1) From the earlier Report (p. 13) "The advantage of introducing formulae early is that they give a clear idea of the main trend of the subject, viz., the statement in symbolic language of general truths. . . . The formula method is more exacting for the teacher and more difficult for the slower boy." (2) From the *Report on Secondary Modern Schools* (p. 3) ". . . in the child the sense of failure leads to frustration and distaste for the artificial culture of the schools." (3) From the *Report on Preparatory School Mathematics*, "Confidence and the satisfaction of getting sums right are very valuable to a beginner." There has been a change of emphasis away from the subject and towards the pupil.

This shift in emphasis, which makes the pupil's enjoyment of success one of the main aims in early lessons, must be allowed for in any readjustment of topics to be taught.

The investigation described below was carried out with the idea of testing various types of preparatory activity, in order to discover, if possible, which of them most effectively subserved the main aim.

Character of the investigation.

The observations were recorded by the teacher of a preparatory form during a period of six months covering the autumn and spring terms. The algebra lessons, each lasting 30 to 40 minutes, and given usually in the middle of the morning session, are, in the record, numbered serially 1 to 42. All errors in the written work were noted and roughly classified at the time, the difficulties occurring in any lesson usually suggesting the procedure to be adopted in the lesson following. The fuller classification, as set out in the Tables, was not undertaken until after the Easter examination.

The class consisted of nineteen girls with an age-range on the 31st August of 9; 3 to 12; 5, ten of the girls being between 10 and 11. Not all of them were beginners: four of the older girls had already spent a year in the form

and had failed to earn their remove, not so much through want of intelligence as through overcrowding in the main school; five were newcomers, from neighbouring primary schools, their school records varying from "fairly good" to "very good"; the remaining ten, including the best pupils, were there by way of routine promotion. It cannot be claimed that the class was a representative sample of a group of beginners, but there was nothing unique or extravagant in its make-up. In the second term epidemics led to many absences. This state of affairs is not unusual, and most teachers of ten-year-olds have to contend with it.

Exercises omitted from the record.

All the written exercises have been classified, with the following exceptions:

(*a*) Exercises done at home.

(*b*) Communal exercises and work done by pairs or small groups, or by children at the blackboard.

(*c*) Work done in Lessons 1 to 4. These exercises were taken from a textbook which was set aside when it proved unsuitable.

(*d*) Work done in Lessons 40 to 42. These lessons were devoted to a scoring game, with signed numbers on a scoring-line; very little manipulation.

(*e*) A few exercises which defied analysis into unit-operations with a definite number right or wrong.

TABLE I (No. of unit-operations)

	A numerical.	B substitution.	C reversing steps.	D axiom of equality.	E situation-problems.	All types.
R	5195	2440	1987	4775	682	15079
W	382	490	388	539	387	2186
Total	5577	2930	2375	5314	1069	17265
% *R*	93·2	83·3	83·7	89·9	63·8	87·3

The recorded exercises.

The thirty-three lessons from which the records are taken yield the results classified in Table I. Some details of the lessons are given next: first, of the more successful lessons (Categories A to D), then those of Category E, in which failure was general. *Column A.* Numerical exercises, set primarily to give practice in the interpretation of expressions and to develop the ability to treat an expression as a psychological unit.

Examples: (1) Pairing equivalent expressions or pairing an expression with a number.

(2) Constructing equivalent expressions, *e.g.* "Make a sum equal to a product."

(3) Similar exercises, with an assigned number appearing once on each side.

(4) Filling gaps, *e.g.* "Repeat the 4 or the 5"

$$\wedge + \wedge = 4 \times 5$$

(5) Evaluating expressions introducing the sixteen possible sequences of two operations chosen from the four fundamental types, *e.g.*

$7+8-2$, $15-(4-2)$, $\frac{10}{1}-4$, $\frac{28}{12-5}$. Integers and very simple fractions only.

(6) Similar to (5) with reversal of one or two steps to find n, "the secret number" *e.g.*

Forward	*Backward*
$9+6-2=$	$n+8-3=16$; $n=$
$4(7+3)=$	$5(n+2)=45$; $n=$

Forward exercises only in Column A, backward exercises in Column C.

(7) From words to mathematical symbols, translating verbs, *e.g.* "Add 7 and 3 and multiply your result by 4."
(*a*) Expression $4(7+3)$ (*b*) Answer 40
(Constructing the expression proved more difficult than calculating the answer.)

(8) From words to mathematical symbols, translating nouns, *e.g.* "Find the difference between 20 and the product of 6 and 3."

(9) Effect of introducing brackets, *e.g.* "Compare $12-7-2$ with $12-(7-2)$."

Column B. Substitution exercises ; the expressions introduce only one letter (usually n) and express one or two operations only. They include the substitutions required in verifying equations.

Column C. Finding the unknown number by reversing one or two operations. In some cases the children constructed the equation from a relation given in words, then solved it ; in others they constructed equations after a set pattern, introducing their own "secret numbers", and passed the equations (written on slips of paper) to other pupils for solution. A popular type of lesson giving the brighter pupils scope for setting one another harder questions.

Column D. Equations solved by applying one of the axioms of equality. The simple type studied requires not more than four applications of the axiom : if equals be taken from equals the remainders are equal. Examples :— $5n+8=7n$; $4n+5=17+2n$. Introduction was by way of a balance, with blackboard drawings, certain square packets, each weighing n ounces, making their appearance alongside round "weights" of 1 ounce. Removal of weights or packets from the balancing pans was related to the relevant changes in the expressions. The brighter girls found no difficulty in formulating their own rules for finding n, and soon gave up the practice of visualising the balancing pans. The others (over half the class) needed considerable help in making this advance. The children called the terms "n-numbers" and "plain-numbers", and decided that, if the R.H.S. happened to be a plain number, it was easier to work by reversing the steps (undoing the expression) than by taking from both sides. Only the brighter girls enjoyed verifying their solutions.

Column E. Use of algebraic symbols in constructing (or selecting) expressions referring to specified real-life situations. The problems assessed in the record involve only one operation and introduce not more than one letter. Example : "There are 24 pupils in a class ; if n are absent, how many are present?" These exercises are described as "referred-symbolic" in contrast to the "free-symbolic" exercises of categories A to D.

Mean scores related to criterion of success.

It will be observed that the ground covered is very much less extensive than is usual for the first six months of a beginners' course. Nevertheless, interest

was sustained throughout and the lessons were enjoyed by most of the girls. How far the enjoyment correlated with the good marks obtained in the first four categories, it is difficult to say ; the five lessons on situation-problems (numbers 5, 7, 8, 17 and 27), with their high proportion of failures, appeared to be enjoyed also. Special care was taken to make these lessons palatable by introducing into them teaching devices which had proved popular with the children. Thus in Lessons 17 and 27, no attempt was made to enforce isolated study. The two lessons took the following course : after a short spell at oral work, the questions, written on cards, were distributed about the room for the children to use as they needed them, each girl writing her initials on the back of a card when she had dealt with it. The answers were chosen from suggested expressions on the blackboard and written on paper, to be corrected at a later stage in the lesson. This kind of lesson automatically took on a characteristic rhythm, quiet periods of concentrated thought alternating with intervals of excited movement while the cards were being exchanged. An element of competition entered in, especially among the more intelligent and ambitious girls and there was the feeling of tension which one associates with a strenuous game.

When the answers were corrected, however, remarks conveying disappointment were common. It was clear that the enjoyment was due to the manner in which the lesson was conducted rather than to any interest inherent in the subject matter.

If success at lessons is to be judged by the sense of satisfaction in achievement experienced by the child, the usual pass-mark is not a suitable criterion; 40% may " satisfy the examiner ", it does not satisfy the child. The mark chosen as a standard must be a mark below which the normal girl would herself consider that she had not done well. It is suggested that 80% of success in these very simple exercises is a suitable value.

TABLE II.

giving ages at beginning of course, mental ratios where known, and scores of 19 girls.

Pupil	D	G	W	A	T	C	N	Y	P	J
Age	10 ; 5	10 ; 5	10;10	10 ; 9	10 ; 5	10 ; 7	10 ; 1	11 ; 0	10 ; 5	9 ; 3
MR*	140	121	122	121	117	120	115	123	124	—
% Score for free exercises	99	99	96	95	94	81	85	92	90	87
% Score for ref. exercises	89	92	87	58	69	59	41	71	57	70

Pupil	M	K	E	S	H	R	F	B	L
Age	11 ; 0	11 ; 11	11 ; 7	12 ; 4	12 ; 2	12 ; 4	12 ; 5	11 ; 10	10 ; 8
MR*	—	—	—	—	—	—	—	—	—
% Score for free exercises	74	85	95	92	80	86	86	92	75
% Score for ref. exercises	66	50	64	56	57	49	55	77	54

* Northern Test of Educability. T. P. Tomlinson.

Table II, in which no context-problem introducing more than one operation is assessed, illustrates the superiority of the " free " exercises. The difference in satisfaction-value between the two types of exercise is significant : in the free exercises, 17 girls reach satisfaction-level, the average score is 88%, with a standard deviation of 7·2 ; in the situation-exercises only 3 girls reach the level and the mean score is only 64%, with standard deviation 13·7.

Analysis of errors in situation-problems.

A detailed analysis of the 387 errors made in answering the situation-problems (Col. E, Table I) has been undertaken, with the following summarised results :

(1) Problems instancing symmetrical processes yielded 77% correct answers, as against 56% where the processes were non-symmetrical, but the incidence of difficult questions was less clear-cut than this result might imply.

(2) In the cases where pupils chose wrong processes, most errors were of the form (*a*) operation substituted for its inverse, or (*b*) multiplication confused with addition or division with subtraction.

Of the 93 inverse-operation errors, 42 occurred in answers to problems in which the situation presented a comparison, stated as difference or as ratio. The wording of the question was shown to be a major cause of trouble, cases where the relatum was given and the referent sought proving easier than those in which the comparison was expressed in the converse sense. This finding is in agreement with observations by Piaget* and other child-psychologists whose work on children's judgments of meaning has a bearing on this point.

The errors due to confusion between primary and derived processes (or between the relevant symbolic devices) were less numerous, but they persisted to the end of the six-months' course and were indeed rather more prevalent in the later lessons. In the answers to questions on addition and subtraction, 11% of the 381 attempts were in error through children writing ab instead of $a+b$ or $\frac{a}{b}$ for $a-b$; in the reverse direction, 6·5% of the 440 attempts to answer questions on multiplication and division were in error.

(3) There were many errors due to reversing the order of the symbols when the situation had been correctly perceived or deduced, and classified as calling for subtraction or division. By a special teaching device it was made possible to discriminate between trouble in classifying the problem and trouble in selecting the appropriate symbolic expression. The results point to the probability that, after the children had analysed the situation intelligently, they resorted to random answering when choosing between $a-b$ and $b-a$, or between $\frac{a}{b}$ and $\frac{b}{a}$.

Subtraction and the partition aspect of division yielded comparable results : correct choice of process in 83% of the 176 attempts, with only 53% correct answers. Questions on quotition and those introducing implicitly the notion of ratio were answered worst, with only 46% of the 178 attempts correct. An example showing the difficulty of these apparently simple exercises is given : (Quotition) Lesson 8, No. 5 : 20 cakes are to be put on plates, n cakes on each plate. How many plates will be needed? Result : $12R$, $7W$. (Ratio) Jan. test, No. 3 : Mrs. X, is n times as old as Mary. If Mrs. X. is 30, how old is Mary? Result : $6R$, $13W$.

(4) There was evidence, chiefly in answers to examination questions, of resistance to the use of letters to replace numbers. In the tests, conditions were more stringent than in the routine lessons, in which the children were

* *Language and Thought of the Child.* (1926). J. Piaget. Ch. 14.

helped by the provision of a set of four, five or six suggested answers to each problem, the child being asked to choose the correct expression from the set. No such help was given in the four tests.

To the weakest pupils it seemed that answers in literal form had not the "feel" of answers; arbitrary numbers were substituted for n in order to bring the situation into a context where the child could cope with it with the confidence born of familiarity. Thus, the ratio question quoted above produced seven numerical answers (out of nineteen attempts); five of the children simply substituted 3 for n, giving Mary's age as 10, which no doubt seemed to them to be a reasonable state of affairs. A definite number apparently helps such children to visualise the situation presented, they infer the correct relationship between the numbers contributing to it, but no general pattern of the relation comes to their minds; the n is a purposeless intruder.

General conclusions.

The errors discussed give some indication of the "set" of the children's minds. The weakest pupils rejected the teaching almost entirely. The moderately good pupils enjoyed what may be described as "fun with symbols" which brought success to their efforts and led to their asking questions foreshadowing continued interest. They showed characteristic reactions to situations beyond their powers: questions framed in the prosaic contexts of everyday life produced random answers, appearing to create a dilemma to escape from rather than a challenge to be faced. A small minority of capable girls enjoyed the satisfaction of success in all the topics studied.

The problem of devising a suitable syllabus for young pupils of medium ability is clearly not solved by making mathematics pedestrian.

E. M. R.

1743. Bantam-weight champion, Tommy Nicholls stood on his head for two minutes in London yesterday to get his weight down for the amateur boxing international between Britain and Italy at Wembley last night. Nicholls should have been 8 st. 7 lb. but was 6 ozs. overweight when he went on the scales for the first time. He was still overweight after exercising. So Captain J. D. Bruen (R.A.O.C.), an ex-Sandhurst cadet and manager of the Army team, said: "Go stand on your head." Nicholls did this and made the weight at the third attempt. Captain Bruen explained: "When I was at school we did P.T., and one of the exercises the instructor was fond of was this. Four of us would have to put our hands on a colleague's head and press down for a few moments. Then one of us would lift him up, with no difficulty at all, with one hand. I remembered this when I took over the R.A.O.C. boxing. By reversing the process, making a fellow stand on his head, you can take off several ounces. The reason I believe is that the blood rushes to the head as he stands on his head, and if he is put on the scales immediately, then the blood moving back down his body has less weight. Of course, if you delay putting him on the scales then he will weigh his proper weight. But as the rules of boxing say that at 1 p.m. when the weigh-in takes place he must be a certain weight it has to be done a few minutes beforehand only." Said Nicholls: "I never saw anything like it."—*Daily Express*, November 15, 1951. [Per Mr. A. R. Pargeter.]

1744. The Board of Trade is to allow manufacturers to reduce the matches in a standard (2d.) box from an average of fifty to one of 47.... Boxes of more than fifty matches or fewer than 47 must not be sold.—*Manchester Guardian*, August 22, 1951. [Per Miss M. Hartley.]

EUCLIDEAN GEOMETRY AND THE RIGID MOTION GROUP.

By R. L. Goodstein.

There is no doubt that Professor D. E. Littlewood's stimulating and instructive article * on the foundations of Euclidean Geometry will be widely read by teachers of geometry both in the Schools and in the Universities. It is almost equally certain that many of these readers, unfamiliar in the main with the language of transformation groups, will be in danger of being misled by an ambiguous usage of the term *motion*.

If it proves to be the case that no reader has in fact been misled then this note will harm no one but the critic himself, and I venture to hope that Professor Littlewood will accept this as evidence of my good intentions.

I start with the categorical affirmation that the motions of rigid bodies, be they rulers, compasses, set squares or cut-out fragments of cardboard, however important they may be in geometrical drawing, engineering or casting a horoscope, are as irrelevant to geometry as colour, taste, sound or any other phenomenon in the physical world. The triangle of geometry is a concept like number or proposition, not a physical object like a printed page, and the figures in a geometry text book constitute just one of the notations in which the theorems may be expressed, not the subject matter of these theorems.

To say that two line segments in Euclidean Geometry are proved equal by moving a rigid body from coincidence with the one into coincidence with the other or that two triangles are proved congruent by moving a cut-out piece of cardboard from coincidence with the first to coincidence with the second is to confuse mathematics with its application. What are the triangles with which the card coincides? Marks on paper? What degree of accuracy of coincidence shall we accept? With what object, one may ask, is a geometry established on a foundation of axioms and rules of derivation if measurement may take the place of proof? Or, to look at it from another standpoint, how many right-angled triangles must we measure before we may consider that Pythagoras' Theorem is established? Surely geometry, like the whole of mathematics, is a deductive system not an experimental science.

The thesis which Klein published at Erlangen in 1872, that a geometry is a system of definitions and theorems which express properties invariant under a given group of transformations, leads to a characterisation of Euclidean Geometry as the geometry of the group of all projective collineations which leave invariant an arbitrary but fixed involution on the line at infinity; this group of transformations is the same as the so-called "rigid motion" group, that is, the group of transformations which leaves invariant the distance between two points. This is not just a happy accident but the very reason why the geometry of the group which leaves invariant the fundamental involution has the same content as the geometry which Euclid constructed on a basis of postulates and axioms. What, however, is the relationship between the rigid motion group of transformations and the actual motion of a rigid body? This question brings us the crux of our inquiry, and the answer to the question is simply that there is *no logical relationship whatever*. To justify this reply we set up a model of Euclidean Geometry as follows:

We choose some base line l_0 in a real projective plane, and an elliptic involution $\mathfrak{I}$ on l. A pair of lines which meet on l_0 are said to be parallel, and a pair which meet l_0 in corresponding points of $\mathfrak{I}$ are said to be perpendicular. Given any point P of the plane, and a line l, the point P', such that PP' is perpendicular to l, and meets l, l_0 in points which harmonically separate P and

* *Math. Gazette*, XXXIV (December, 1950), p. 272.

P', is called the orthogonal reflection of P in l. The result of two consecutive orthogonal reflections is called a rigid motion.

If the points P', Q' are the transforms of P, Q and the lines $P'Q'$, $P'R'$ the transforms of PQ, PR under a rigid motion, then the point pair $P'Q'$ is said to be congruent with the pair PQ and the line pair $P'Q'$, $P'R'$ is congruent with PQ, PR.

It can readily be shown that all the familiar propositions of Euclidean Geometry are provable in this model. For instance

(i) one and only one parallel can be drawn to a given line through a given point ;

(ii) if $PQRS$ is a parallelogram then the point pairs PQ, RS are congruent, and so too the line pairs PQ, PS ; RQ, RS ;

(iii) if the triads ABC, $A'B'C'$ are such that the pairs (AB, AC), $(A'B', A'C')$; AB, $A'B'$ and AC, $A'C'$ are congruent, then so are BC, $B'C'$.

(iv) if a conic with respect to which $\mathfrak{I}$ is an involution of conjugate points is called a circle then there is one and and only one circle through any three points, and any two radii of the circle (rays from the pole of l_0) are congruent.

If we now make a representation of this model on a drawing board, taking any line other than l_0 for the line at infinity, we find that we can no longer cover two congruent triangles in turn by the same physical body, nor have the representations of congruent point pairs the same separation measured by a ruler.

When we say in geometry that a triangle ABC can be brought into coincidence with another triangle $A'B'C'$ by a rigid motion, we are using the term *motion* metaphorically. We are not talking about physical movement but are asserting the existence of a certain configuration, in this case a pair of lines l_1, l_2 and a triangle ABC such that $A^*B^*C^*$ is the orthogonal reflection of ABC in l_1, and $A'B'C'$ is the orthogonal reflection of $A^*B^*C^*$ in l_2. Russell rightly described Euclid's proof by superposition of the congruence of two triangles in the case of two sides and the included angle as a tissue of nonsense, because Euclid did not establish the existence of a " rigid motion " configuration, did not, in fact, even attempt to define the " rigid motion " of a triangle. And the " proof " which Littlewood proposes is not mathematics at all.

By all means let children *approach* geometry by way of geometrical drawing and the movement of paper triangles, but do not confuse the traveller's coach with his destination. However much we may use the language of the physical world, geometry is a self-contained logical structure, not an empirical science, neither validating nor invalidated by any natural phenomenon, and owing nothing to the existence of physical bodies or the possibility of motion but the accident of its history.

R. L. G.

1745. We have just supplied them with details of 20 consignments for London (of less than 10 cwts. each) picked at random. One consignment was delivered next day, seven within two days and four within three days.

The rest took four, five or six days to arrive. Normal delivery to London under private enterprise was always the next day, so it seems that nationalised road service is anything from 50 to 600 per cent. slower.—Letter in *News-Chronicle*, September 29, 1951. [Per Mr. P. Vermes.]

MATHEMATICAL NOTES

2334. *The Name of the Game of Nim.**

The game usually called NIM is very well-known both to mathematicians and to professional gamblers and for the same reason to both: while appearing to be a game of chance, it has, in fact, a perfect theory of winning. In a standard discussion of the theory of this game—that at G. H. Hardy and E. M. Wright, *An introduction to the Theory of Numbers* (2nd ed.), p. 116 ff.—the game is referred to as "The Chinese Game of Nim". A reference such as this might well lead—and indeed has often led—people to think that *Nim* was the Chinese name for the game. A priori, this would seem improbable (for, save in the Cantonese dialect, Chinese words do not end in *-m*) and is, in fact, not the case.

Two citations are relevant here:

(1) W. Ahrens, *Mathematische unterhaltungen und spiele* (2nd ed., 1910)† i, 72: "Der ursprung des spiels ist unbekannt. In Amerika wird es auf einigen Colleges bisweilen dort auch wohl auf Jahrmärkten gespielt; es soll aber auch in Deutschland schon seit Jahrzehnten bekannt sein‡. Auch in China soll es gespielt werden und möglicherweise ist dies sein Ursprungsland. Jedenfalls trägt es einen chinesischen Namen 'Fan-Tan'. Da man jedoch mit demselben Namen auch ein ganz anderes, bei Chinesen beliebtes Spiel bezeichnet, das ein reines Glücksspiel ist, so haben wir hier den . . . von Chs. L. Bouton vorgeschlagenen Namen akzeptiert."

(2) C. L. Bouton, "Nim, a game with a complete mathematical theory," *Annals of Mathematics*, II. iii [1901–2], 35: "The writer has not been able to discover much concerning its history, although certain forms of it seem to be played at a number of American Colleges, and at some of the American fairs. It has been called Fan-Tan, but as it is not the Chinese game of that name, the name in the title is proposed for it."

Professor Bouton does not tell us why he gave the name *Nim*§ to the game. He died in 1922‖ and it may well be that there are people still living who actually know the answer to this question. It would be pleasant if one such should read this note. But in the absence of evidence of this kind the etymology of the name is for discussion. It may, of course, be that the name is arbitrary. But, in fact, word-coinages which are truly arbitrary, both consciously and sub-consciously, are exceedingly rare**—Dean Swift appears to constitute a notable exception here. It seems therefore more probable that *nim* is not arbitrary but that it is merely the imperative singular of the German verb *nehmen* "to take", semantically a most suitable nomenclature. This form is today spelt *nimm*; it may be that Professor Bouton dropped the final *m*.

* I am very grateful to Professor W. Simon (London) and Professor H. H. Dubs (Oxford) for advice on the Chinese side of the matter.

† It is to this work that Hardy and Wright refer.

‡ But Ahrens does not give the German name of the game, nor have I been able to discover it.

§ The word does not appear either in the *New English Dictionary* or in W. A. Craigie, *A Dictionary of American English*.

‖ He was Assistant Professor of Mathematics at Harvard from 1904 to 1914 when he became Associate Professor, a position he held until his death. (Information kindly given me by the Alumni Records Office of Harvard University.)

** Cf. my remark on Dr. Aavik's coinage *relv* "weapon" (now part of the normal Estonian vocabulary) at *Transactions of the Philological Society* 1938, p. 68, note 2.

With regard to the Chinese name *Fan-Tan*. In Europe, this name is applied to two gambling games not only to one (as stated by Ahrens). The first is well-defined by the *New English Dictionary*, *s.v.*: "*Fan-Tan*.... A Chinese gambling game, in which a number of small coins are placed under a bowl and the players then bet as to what will be the remainder when the pile has been divided by four." This game is very well-known* and is of bad repute. Its bad reputation is, in part, due to a special cause. There is a common form of the game which does not exactly conform to that described by the *New English Dictionary*. In this form the players first make their bets on the remainders 0, 1, 2 and 3 and not until then does the croupier take a handful of small coins (often, cowries or counters) and remove them four by four until the remainder is known. It is widely believed among gamblers that the "good" croupier sees on which remainder the betting is lightest and is so skilled that he is then able to choose (by eye or feel) the number of coins, etc., so as to leave the appropriate remainder—in view of the large number of coins, etc., in the handful, it may be doubted whether this belief can have any foundation in fact. The name *Fan-Tan* is also applied to a card-game of the Pope-Joan type described, for instance, by L. H. Dawson, *Hoyle's Games modernized* (1951 ed.), pp. 218–19†.

It seems that the use of the name *fan-tan* for "nim" is not, in fact, known in China. Professor Dubs plausibly suggests that this use is a non-Chinese misnomer, the confusion merely being due to the fact that both fan-tan and nim are games played with heaps of counters; this explanation would also serve to cover the application of the name *fan-tan* to the game of the Pope-Joan type mentioned above.

ALAN S. C. ROSS.

2335. *Differential equations by vector methods.*

The following two short notes would be more appealing if they could be rounded off. In the first note, having established a necessary condition, it would be nice if we could show that this guarantees the existence of an integrating factor and so proves sufficiency. In the second note we lack a proof that the knowledge of a multiplier and a solution enables a second solution to be found by quadrature.

1. The total differential equation.

The coefficients in the equation

$$P\,dx + Q\,dy + R\,dz = 0 \quad \text{.....(i)}$$

can be used to define a vector field by writing

$$\mathbf{V} = \mathbf{i}P + \mathbf{j}Q + \mathbf{k}R. \quad \text{.....(ii)}$$

The position vector being $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, the equation (i) is equivalent to the scalar product

$$d\mathbf{r} \,.\, \mathbf{V} = 0. \quad \text{.....(iii)}$$

It is possible that $\mathbf{V}$ is the gradient of some scalar; alternatively, $\mathbf{V}$ may or may not be a scalar multiple of a gradient. We take these in order.

If $\mathbf{V} = \nabla\phi$ the equation (iii) becomes $d\mathbf{r} \,.\, \mathbf{V} = d\mathbf{r} \,.\, \nabla\phi \equiv d\phi = 0$ and the equation (i) integrates to $\phi = \text{const.}$ In this case we have, by taking the curl,

$$\nabla \times \mathbf{V} = \nabla \times \nabla\phi \equiv 0.$$

* See H. A. Giles, *A glossary of reference on subjects connected with the Far East* s.v. *Fantan*; S. Couling, *Encyclopaedia Sinica* s.v. *Fantan*; S. Culin *The gambling games of the Chinese in America: Fán t'án ... and Pák kòp piú*.

† This use of the name *Fan-tan* is not mentioned in the *New English Dictionary*—it may be quite recent; the game was formerly called *Play or pay* (a use of this phrase not mentioned in the *New English Dictionary*'s discussion of it s.v. *Play*, v. 17e).

Conversely, if $\nabla \times \mathbf{V} = 0$ an application of Stokes's theorem shows that $\mathbf{V}$ is the gradient of some scalar (Weatherburn, p. 27) and $\mathbf{V} = \nabla\phi$. Hence the necessary and sufficient condition that (i) should be exact is $\nabla \times \mathbf{V} = 0$. Written in the extended form, this is the usual

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

Alternatively, we suppose that the components of $\mathbf{V}$ are merely proportional to the components of some gradient and we write $\mu\mathbf{V} = \nabla\phi$. On taking the curl we have

$$\nabla \times \nabla\phi \equiv 0 = \nabla \times (\mu\mathbf{V}) = \nabla\mu \times \mathbf{V} + \mu\nabla \times \mathbf{V}. \quad \text{.....(iv)}$$

If this be multiplied scalarly by $\mathbf{V}$ we have

$$\mu\mathbf{V} \,.\, \nabla \times \mathbf{V} = \mathbf{V} \times \nabla\mu \,.\, \mathbf{V} \equiv 0. \quad \text{.....(v)}$$

As μ is not zero, we have

$$\mathbf{V} \,.\, \nabla \times \mathbf{V} = 0 \quad \text{.....(vi)}$$

which is usually written

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

Conversely, if the relation (vi) holds, we have, for any scalar μ, the relations

$$\mu(\mathbf{V} \,.\, \nabla \times \mathbf{V}) = 0, \quad \mathbf{V} \times \nabla\mu \,.\, \mathbf{V} \equiv 0.$$

On subtracting and multiplying by μ, we reach

$$(\mu\mathbf{V}) \,.\, \nabla \times (\mu\mathbf{V}) = 0. \quad \text{.....(vii)}$$

Comparison of (vi) and (vii) shows that unless (vi) holds there is no point in trying to find an integrating factor ; none such exists.

2. Jacobi's last multiplier.

If $f(x, y, z) = a$ is a solution of the simultaneous equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}, \quad \text{.....(i)}$$

we have

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} + Z\frac{\partial f}{\partial z} = 0. \quad \text{.....(ii)}$$

Take the denominators of (i) as components of a vector, so that

$$\mathbf{F} = \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z.$$

Also, f is a scalar point-function and (ii) reads $\mathbf{F} \,.\, \nabla f = 0$. Thus $\mathbf{F}$ is perpendicular to the gradient of f ; and if we have a second solution g, the direction of $\mathbf{F}$ is perpendicular to the plane of the gradients of f and g. Hence, there exists a relation of the form $\nabla f \times \nabla g = M\mathbf{F}$ where M is some scalar.

Apply the expansion formula for the divergence of a vector product

$$\nabla \,.\, (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \,.\, \nabla \times \mathbf{v} - \mathbf{v} \,.\, \nabla \times \mathbf{u}$$

and we get $\nabla \,.\, (M\mathbf{F}) = 0$ since $\nabla \times \nabla f \equiv 0$. This is the equation satisfied by the multiplier M, and if we use the expansion formula for the divergence of a scalar-vector product we have $M\nabla \,.\, \mathbf{F} + \mathbf{F} \,.\, \nabla M = 0$. In particular, if $\mathbf{F}$ is solenoidal we are left with $\mathbf{F} \,.\, \nabla M = 0$, of which an obvious solution is $M = 1$, the dynamically important case (Whittaker, p. 280).

When two multipliers M and N are known we have

$$\nabla \,.\, (M\mathbf{F}) = 0 = M\nabla \,.\, \mathbf{F} + \mathbf{F} \,.\, \nabla M,$$
$$\nabla \,.\, (N\mathbf{F}) = 0 = N\nabla \,.\, \mathbf{F} + \mathbf{F} \,.\, \nabla N.$$

Eliminating $\nabla \cdot \mathbf{F}$, we reach

$$\mathbf{F} \cdot \{N\nabla M - M\nabla N\} = 0 = N^2\mathbf{F} \cdot \nabla(M/N).$$

This states that $\mathbf{F}$ is perpendicular to the gradient of M/N; in other words, the ratio of two multipliers is a solution of (i). F. E. RELTON.

2336. *A Special Determinant.*

Let D_n denote the matrix $\left[\binom{n+1}{2j-i}\right]$, having the indicated binomial coefficient as the element in its ith row and the jth column. A. M. Nesbitt stated without proof (see Muir, *History of Determinants*, 1900–1920, p. 316) that the determinant

$$|\, D_n \,| = 2^{\frac{1}{2}n(n+1)}.$$

Muir suggested a proof based on centrosymmetry. We suggest the following, based on the familiar recurrence-relations satisfied by binomial coefficients. To save space we illustrate (like Muir) by the case $n = 4$. First,

$$|\, D_4 \,| = \begin{vmatrix} 5 & 10 & 1 & & \\ 1 & 10 & 5 & & \\ & 5 & 10 & 1 & \\ & 1 & 10 & 5 & \\ & & & & \end{vmatrix} = \begin{vmatrix} 5 & 10 & 1 & & \\ 1 & 10 & 5 & & \\ & 5 & 10 & 1 & \\ & 1 & 10 & 5 & \\ & & 5 & 10 & 1 \end{vmatrix} = \begin{vmatrix} & & & & \\ & 5 & 10 & 1 & \\ & 1 & 10 & 5 & \\ & & 5 & 10 & 1 \\ & & & & \end{vmatrix} = |D'_4|.$$

But, by the recurrences mentioned, we have

$$\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ 1 & -5 & 10 & -10 & 6 \end{bmatrix} \begin{bmatrix} 5 & 10 & 1 & & \\ 1 & 10 & 5 & & \\ & 5 & 10 & 1 & \\ & 1 & 10 & 5 & \\ & & 5 & 10 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 20 & 6 & & \\ 1 & 15 & 15 & 1 & \\ & 6 & 20 & 6 & \\ & 1 & 15 & 15 & 1 \\ & & 6 & 20 & 6 \end{bmatrix},$$

let us say

$$H_4 D'_4 = D_5.$$

Taking determinants and clearing the superdiagonal of $|\, H_4 \,|$ by operations $\mathrm{col}_2 - \mathrm{col}_1$, $\mathrm{col}_3 - \mathrm{col}_2$, ... , $\mathrm{col}_5 - \mathrm{col}_4$, we have

$$|\, D_5 \,| = 2^5 \,|\, D_4 \,|.$$

The general procedure is in all respects analogous. Hence, since $|\, D_1 \,| = 2$, step by step we have

$$D_n = 2 \cdot 2^2 \cdot 2^3 \ldots 2^n = 2^{\frac{1}{2}n(n+1)}.$$

A. C. AITKEN.

2337. *Proofs in elementary geometry.*

Deductive geometry as a recognised part of grammar school syllabuses seems to be on the way out, and may take its place as part of a Sixth Form specialist course. The flood of geometry textbooks for the young may well therefore be lessened, and the quality of such publications may improve. May I appeal to future authors of school geometries to show evidence of a wider reading than the *Elements*? Dr. Lietzmann in *Altes und Neues vom Kreis* and *Der Pythagoreische Lehrsatz* has shown what can be done with two well-known topics, but his efforts seem still unknown in England. The purpose of this note is to commend his proofs of three theorems which in English textbooks still remain in Euclid's rather unsatisfactory style.

The first has been published in Note 2111 of the *Gazette* (XXXIV, No. 307) and need not be repeated. The second, companion to the first, was published in Note 1246 (XXI, No. 244), and so may bear repetition.

1. Given $\angle ACB = \angle ADB = a$, to prove that A, B, C, D are concyclic.

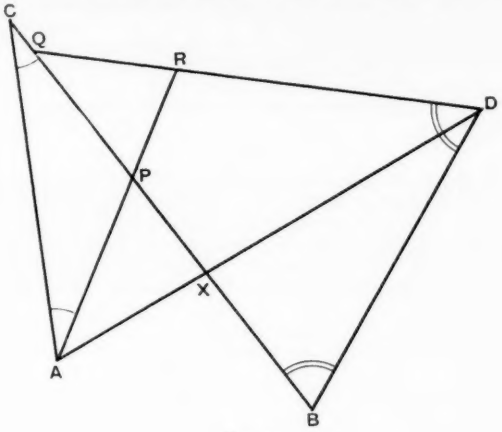


FIG. 1.

The triangles CXA, DXB are equiangular. Let $\angle CAD = b = \angle DBC$, and suppose that $a < b$. Let P be a point on BC such that $\angle PAC = \angle PCA$, and Q on BC such that $\angle QDB = \angle QBD$. Let AP and DQ intersect at R. Then

$$\angle RAD = b - a = \angle RDA,$$

and it follows that RDA is isosceles.

Now, the bisectors of the angles BQD, ARD, RPB meet at a point, Z, say. But as BQD is isosceles, the bisector of $\angle BQD$ bisects BD at right angles, and therefore Z is equidistant from B and D. Similarly Z is equidistant from A and D, and from A and C.

2. Given that in the triangle ABC, $AC^2 = AB^2 + BC^2$, to prove that the angle B is a right angle.

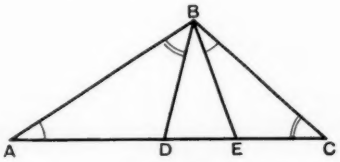


FIG. 2a.

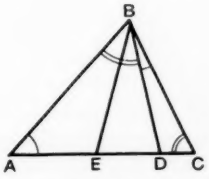


FIG. 2b.

Let $\angle ABD = \angle ACB$ and $\angle CBE = \angle BAC$. Then the triangles ABC, ADB, BEC are similar and hence

$$AB/AD = AC/AB, \quad AC/BC = BC/EC.$$

Thus

$$AB^2 = AC \cdot AD, \qquad BC^2 = AC \cdot EC.$$

Therefore $AB^2 + BC^2 = AC(AD + EC)$ (Fig. 2a)

or $= AC(AD - EC)$ (Fig. 2b).

Therefore, by hypothesis, DE in either case is zero, whence it follows that

$$\angle ABC = \angle BAC + \angle ACB = 1 \text{ right angle.}$$

J. E. BLAMEY.

2338. *Simson's line and its envelope.*

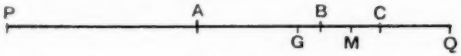


FIG. 1.

If A, B, C are three points on a line (Fig. 1) having coordinates α, β, γ, measured from an origin P on the line, there is a point G, defined by the coordinate $\frac{1}{3}(\alpha+\beta+\gamma)$, whose position is independent of the origin. G can be constructed as the point of trisection nearer to M of the segment AM, where M is the mid-point of BC. The point Q, defined by the coordinate $\frac{1}{2}(\alpha+\beta+\gamma)$, depends on P, and may be constructed by producing PG to $1\frac{1}{2}$ times its own length.

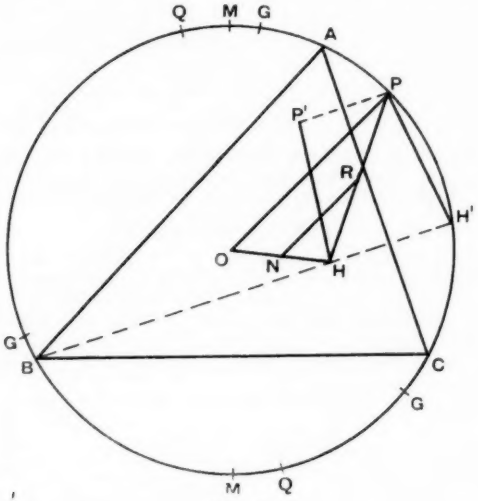


FIG. 2.

A, B, C are three points on a circle (Fig. 2) having coordinates α, β, γ; these being the angles subtended at the centre O by the arcs PA, PB, PC. The coordinates thus represent the directions OA, OB, OC, and may equally well be taken as $\alpha+n\,.\,360°$, $\beta+n'\,.\,360°$, $\gamma+n''\,.\,360°$. There are then two mid-points M of the arcs BC, and their coordinates are $\frac{1}{2}(\beta+\gamma)+n\,.\,180°$; likewise three points G, independent of P, whose coordinates are

$$\tfrac{1}{3}(\alpha+\beta+\gamma)+n\,.\,120°.$$

These may be constructed by trisecting the arcs AM so that $AG = 2GM$. Q has the coordinates $\frac{1}{2}(\alpha+\beta+\gamma)+n\,.\,180°$, and its two positions depend on P. The two points Q may be constructed by producing the arcs PG to $1\frac{1}{2}$ times their own length.

It will be shown that the direction OQ is that of the Simson Line of P with respect to triangle ABC, and that the directions OG are those of the three cusps of the envelope.

Let P' be the image of P in AC, and let H be the orthocentre, with H' its image in AC. Then H' is on the circle, and the angle from $H'P$ to $H'B$ is one of the angles $\frac{1}{2}(\beta + n \cdot 360°)$. By symmetry, the angle from HH' to HP' is the same, and, the direction BH' having the coordinate $\frac{1}{2}(\alpha+\gamma)$, that of HP' is

$$\tfrac{1}{2}(\alpha+\beta+\gamma)+n \cdot 180°.$$

The symmetry of this result shows that the images of P in the three sides of the triangle are collinear with H, the direction being that of OQ.

It follows that the feet of the perpendiculars from P to the sides of the triangle are collinear with the midpoint R of PH, the direction being again that of OQ (Simson's Line). Moreover, if N is the midpoint of OH, $NR = \frac{1}{2}OP$. Therefore R lies on the nine-points circle.

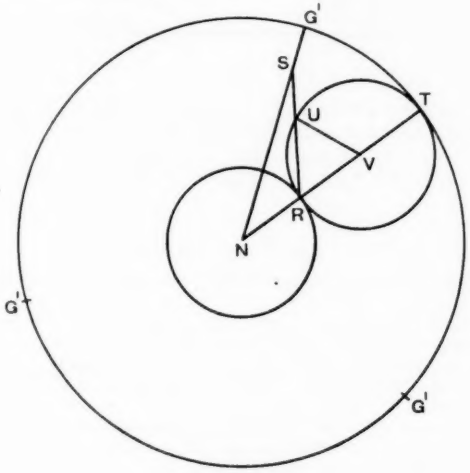


FIG. 3.

Let a circle TG' be drawn (Fig. 3) with centre N and radius $3NR$, NR produced cutting it at T and NG' being parallel to OG. Then the angle from NT to NG' is $\frac{1}{3}(\alpha+\beta+\gamma)+n \cdot 120°$. From R draw RS at an angle

$$\tfrac{1}{2}(\alpha+\beta+\gamma)+m \cdot 180°$$

measured from RT. Then RS is the Simson Line of P.

Draw the circle on RT as diameter, centre V, cutting RS at U. Then the angle from VT to VU is twice that from RT to RU, and hence three times that from NT to NG'. Hence arc TU = arc TG', and the locus of U is a three cusped hypocycloid with cusps at G'. Moreover, angle RUT is a right angle and T is the instantaneous centre of the rolling circle; therefore RS, the Simson Line, touches the hypocycloid at U. The hypocycloid is thus the envelope of the Simson Line.

E. H. L

2339. *On the approximation $\theta = \tan\theta$, when θ is small.*

We are taught at school, and handbooks of formulae encourage the fact, that when θ is sufficiently small $\tan\theta$ may be replaced by θ, and *vice versa*, whichever is the more convenient for our particular application. As far as I can find, however, no warning is given that this result may produce quite incorrect answers if not used with some care.

Consider its use in obtaining an approximate formula for the area of half of a circular segment, in terms of the height h and the base c, when the angle of the sector is small.

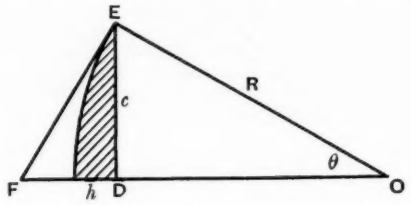


FIG.

The shaded area is $\frac{1}{2}R^2\theta - \frac{1}{2}c(R-h)$, and we see that $\tan\theta = c/(R-h)$. Eliminating R and taking θ small enough to be replaced by $\tan\theta$, the area is

$$\tfrac{1}{2}\left(\frac{c^2+h^2}{2h}\right)^2 \cdot \frac{2hc}{c^2-h^2} - \tfrac{1}{2}\,\frac{c\,.\,(c^2-h^2)}{2h}$$

which reduces to

$$\frac{hc^2}{c^2-h^2} = hc\,.\,\frac{c^2}{c^2-h^2} > hc,$$

that is, the segmental area is greater than the area of the surrounding rectangle of sides h, c.

All we have done, of course, by our " approximation " is to find the (exact) area of the triangle DEF, by a rather long route, which being long and coupled with our attention on the segment has produced an erroneous result ; but I doubt if the error would be readily apparent without a fairly close examination.

In the application in which this error occurred, it was only noticed that something was wrong with $hc^2/(c^2-h^2)$ by means of the fact that since θ is small, then for fixed c, h/c would be small, and the formula then gives the approximation hc, whereas one would expect $\frac{1}{2}hc$.

An error of some two to one is hardly an approximation.

C. C. PUCKETTE.

2340. $\int_a^b x^m dx$: *ab initio evaluation.*

It is well known that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

where, as $n\to\infty$, each $(x_r - x_{r-1})$ tends to zero and ξ_r is any point in the interval (x_{r-1}, x_r), these intervals being such that

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Now we have, for m a positive integer,

$$(x^{m+1} - y^{m+1})/(x-y) = x^m + x^{m-1}y + \ldots + xy^{m-1} + y^m, \quad \ldots\ldots\ldots\ldots(i)$$

which can be expressed as

$$a_0x^m + a_1\left(\frac{x+y}{2}\right)^m + a_2\left(\frac{x+2y}{3}\right)^m + \ldots + a_m\left(\frac{x+my}{m+1}\right)^m, \quad \ldots\ldots\ldots\text{(ii)}$$

where $a_0, a_1, \ldots, a_m$ can be determined by equating coefficients of $x^m, x^{m-1}y, \ldots$ and solving the $m+1$ simultaneous linear equations in the unknowns

$$a_0, a_1, \ldots, a_m.$$

Putting $x = y$, we immediately get the identity

$$m+1 = a_0 + a_1 + \ldots + a_m. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

In (a), put $\xi_r = (x_r + \lambda x_{r-1})/(\lambda+1)$, which is a point in the interval (x_{r-1}, x_r). Then

$$\int_a^b x^m dx = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{x_r + \lambda x_{r-1}}{\lambda+1}\right)^m (x_r - x_{r-1}). \quad \ldots\ldots\ldots\ldots\text{(iv)}$$

Multiply this by $a_0, a_1, \ldots, a_m$ in turn and add, taking $\lambda = 0, 1, \ldots, m$ respectively.

$$(a_0 + a_1 + \ldots + a_m)\int_a^b x^m dx$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left\{a_0(x_r)^m + a_1\left(\frac{x_r + x_{r-1}}{2}\right) + \ldots + a_m\left(\frac{x_r + mx_{r-1}}{m+1}\right)^m\right\}(x_r - x_{r-1})$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{{x_r}^{m+1} - {x_{r-1}}^{m+1}}{x_r - x_{r-1}}\right)(x_r - x_{r-1}),$$

by virtue of (i) and (ii).

Hence

$$(m+1)\int_a^b x^m dx = \lim_{n\to\infty} \sum_{r=1}^{n} ({x_r}^{m+1} - {x_{r-1}}^{m+1})$$

$$= b^{m+1} - a^{m+1}$$

and so

$$\int_a^b x^m dx = (b^{m+1} - a^{m+1})/(m+1).$$

Other cases can be deduced by a change of variable: for instance, for a rational index p/q, write $x = y^q$. A. K. SRINIVASAN.

2341. *Single-valued series of multi-valued functions.*

That a series of single-valued terms may define a function which is multi-valued is one of the surprising facts revealed by elementary analysis. Not quite so remarkable and certainly much simpler to demonstrate is the existence of series of multi-valued terms whose sums are single-valued, For, let $f(z)$ be any multi-valued function and let $f_1(z), f_2(z), \ldots$ be all its branches. Then, provided the series $\sum_{k=1}^{\infty} f_k(z)$ converges absolutely over some region of the z-plane, it defines a single-valued function, for if z describes a closed path in its plane, the only effect this will have is to alter the order in which the terms occur in the series leaving its sum unchanged.

In this note we shall discuss the case when $f(z) = g(\log z)$, g being a single-valued function of its argument. Then, if $\operatorname{Log} z$ denotes the principal value of $\log z$ so that $\log z = \operatorname{Log} z + 2\pi ki$, we have to consider the single-valued function defined by the series

$$G(z) = \sum_{k=-\infty}^{\infty} g(\operatorname{Log} z + 2\pi ki). \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(1)}$$

In all the examples we shall give, g will be of the form

$$g(w) = e^{-\mu w} R(w), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where $0 \leqslant \mu < 1$ and R is a rational function whose denominator is of degree at least two greater than that of its numerator. Under these circumstances, if $R(w)$ possesses a pole of the mth order at $w = a$, one term of the series (1) and one term only will possess a pole of the same order at $z = e^a$. Removing all such terms corresponding to the poles of R from (1), we are left with a series of terms which are regular over any finite region of the z-plane not including the origin. This series is absolutely and uniformly convergent over any such region in view of the inequality

$$| g(\mathrm{Log}\, z + 2\pi k i) | \leqslant A/k^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where A is independent of z, and hence defines a function regular throughout such a region. $G(z)$ is therefore a single-valued analytic function which, apart from a finite number of poles corresponding to those of R, can only possess isolated singularities at the origin and at infinity. In fact, we shall now show that G is regular at both these points, and it will then follow that G is rational.

Let γ be a small circle in the z-plane centre the origin, and radius ρ. The series for $G(z)$ being uniformly convergent over γ we have

$$\int_\gamma z^n G(z)\, dz = \sum_{k=-\infty}^{\infty} \int_\gamma z^n g(\mathrm{Log}\, z + 2\pi k i)\, dz, \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where n is a positive integer or zero. For a given k and sufficiently small ρ we may write

$$\left| \int_\gamma z^n g(\mathrm{Log}\, z + 2\pi k i)\, dz \right| \leqslant B \rho^{n+1-\mu}, \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where B is independent of ρ. From (5) we deduce that

$$\lim_{\rho \to 0} \int_\gamma z^n g(\mathrm{Log}\, z + 2\pi k i)\, dz = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

and since this integral is independent of small ρ

$$\int_\gamma z^n g(\mathrm{Log}\, z + 2\pi k i)\, dz = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Substituting in (4), we obtain

$$\int_\gamma z^n G(z)\, dz = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

so that the Laurent expansion of G about the origin contains no inverse powers of z and G is accordingly regular at $z = 0$ if $G(0)$ be defined suitably.

From (1)

$$\begin{aligned} G(z^{-1}) &= \sum_{k=-\infty}^{\infty} g(-\mathrm{Log}\, z + 2\pi k i) \\ &= \sum_{k=-\infty}^{\infty} g(-\mathrm{Log}\, z - 2\pi k i) \\ &= \sum_{k=-\infty}^{\infty} h(\mathrm{Log}\, z + 2\pi k i), \quad \ldots\ldots\ldots\ldots\ldots\ldots (9) \end{aligned}$$

where

$$h(w) = g(-w) = e^{\mu w} S(w) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

and S is rational. We can now prove exactly as above, replacing $-\mu$ by μ, that the series (9) represents a function regular at $z=0$. $G(z)$ is therefore regular at infinity. Further, in this case the equation corresponding to (7) is also true with $n=-1$, so that $G(\infty)=0$.

We have now collected sufficient information about $G(z)$ to permit its calculation when R is any rational function.

Thus, suppose $\mu=0$ and

$$R(w)=\frac{1}{w^2-\alpha^2}. \quad \text{.....(11)}$$

Near $z=e^{\alpha}$, $G(z)$ behaves like $1/(\log^2 z-\alpha^2)$, where we take $\log z=\alpha$ at $z=e^{\alpha}$. Its principal part in the neighbourhood of this point is accordingly

$$1/2\alpha(e^{-\alpha}z-1).$$

Similarly, in the neighbourhood of $z=e^{-\alpha}$, $G(z)$ possess a principal part

$$-1/2\alpha(e^{\alpha}z-1).$$

Since $G(\infty)=0$ we must have

$$G(z)=\frac{z\sinh\alpha}{\alpha(z^2-2z\cosh\alpha+1)}. \quad \text{.....(12)}$$

Putting $z=1$, $\alpha=2ix$ in (12) we obtain after some manipulation

$$\sum_{k=-\infty}^{\infty}\frac{1}{x^2-k^2\pi^2}=\frac{\cot x}{x}, \quad \text{.....(13)}$$

a formula which is obviously equivalent to the well-known expansion of $\cot x$, *viz.*

$$\cot x=\frac{1}{x}+2x\sum_{k=1}^{\infty}\frac{1}{x^2-k^2\pi^2}. \quad \text{.....(14)}$$

If instead we make the substitutions $z=-1$, $\alpha=2ix$, we obtain the corresponding expansion of $\tan x$, *viz.*

$$\tan x=x\sum_{k=0}^{\infty}\frac{1}{(k+\tfrac{1}{2})^2\pi^2-x^2}. \quad \text{.....(15)}$$

Taking $\mu=\frac{1}{2}$ and $R(w)$ as at (11), we find by the method described above that

$$G(z)=\frac{(z+1)\sinh\frac{1}{2}\alpha}{\alpha(z^2-2z\cosh\alpha+1)}. \quad \text{.....(16)}$$

From which, by the substitutions $z=1$, $\alpha=2ix$, we obtain the expansion of $\operatorname{cosec} x$, *viz.*

$$\operatorname{cosec} x=\frac{1}{x}+2x\sum_{k=1}^{\infty}\frac{(-1)^k}{x^2-k^2\pi^2}. \quad \text{.....(17)}$$

By a suitable choice of μ and R it is evident that we may obtain the other well-known expansions of the same class.

We may also sum all series of the type

$$\sum_{k=-\infty}^{\infty}r(k)\begin{matrix}\cos\\ \sin\end{matrix}2k\pi\theta,$$

where r is rational in k and $O(1/k^2)$ for large k, by this method. For if we put $\mu=\theta$ and choose R so that $R(ki)=r(k)$, by summing the resulting series (1) and then setting $z=1$, we easily extract the result required. The result so obtained is valid for $0\leqslant\theta<1$, but is easily extended to other values of θ.

For example, to sum the series $\sum_{k=1}^{\infty} \cos 2\pi k\theta/k^2$, we take $R(w) = (i/w)^2$ and with $\mu = \theta$ obtain

$$G(z) = -\frac{\theta + (1-\theta)z}{(1-z)^2}. \quad \text{.....(19)}$$

Transferring the term in the series for $G(z)$ with a pole at $z = 1$ to the right-hand side of equation (19) and then letting $z \to 1$, we obtain

$$\frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{\cos 2k\pi\theta}{k^2} = \lim_{z\to 1} \left\{ \frac{1}{e^{\theta \log z} \log^2 z} - \frac{\theta + (1-\theta)z}{(1-z)^2} \right\}, \quad \text{.....(20)}$$

the branch of $\log z$ intended being that which tends to zero as $z \to 1$. From (20) we now obtain the result required, *viz.*

$$\sum_{k=1}^{\infty} \frac{\cos 2k\pi\theta}{k^2} = \pi^2(\theta^2 - \theta + \tfrac{1}{6}), \quad \text{.....(21)}$$

a result which may be verified by Fourier expansion. D. F. LAWDEN.

2342. *The penny problem.*

Mathematical Note No. 1931 (October 1946) gives a very neat solution to the problem of identifying a defective penny among a collection of M pennies, using a common balance without weights. If n operations of weighing are allowed, then $3 \leqslant M \leqslant \frac{1}{2}(3^n - 3)$.

The solution involves labelling the pennies in the ternary scale. This method lends itself to a very effective demonstration of the problem, if, instead of using numbers for the labels, we use patterns of different shape, colour etc. For example, for $n = 3$, $M = 12$, we may code the numbers as follows. For the 1st digit of the ternary number, 0 is a circle, 1 is a hexagon, 2 is a square. For the 2nd digit, 0 is red, 1 is yellow, 2 is blue. For the 3rd digit, 0 is a white centre, 1 has no centre, 2 is a black centre. Thus label 120 becomes a blue hexagon with a white centre. The patterns may be drawn on a card, and you ask your victim to choose a label (without telling you) and decide if it is heavy or light. You then tell him your weighing operations, *e.g.* weigh the circles against the squares, and he supplies the results, from which you identify the faulty coin. The method is, of course, much more effective for larger values of n, and the absence of numbers makes it very puzzling to the uninitiated.

I have made a rug to demonstrate the case of $n = 5$, $M = 120$, which contains 8 rows of 15 labels each. If the labels are arranged in the order of the ternary numbers, the pattern of the rug assumes a certain amount of regularity, and to avoid this I have mixed them up, but I have kept each group of three cyclic permutations together, so as to be able to use the method for any number of coins less than 120. When the weighings have been completed, you can describe the pattern of the faulty coin, but it is more effective to be able to point to it. Owing to the mixing up of the labels, it is necessary to have a chart to find the position of the faulty coin. The rug is rather startling in design, but causes a lot of entertainment.

I believe this problem originated during the late war, and it very quickly went the rounds of the Government and Services scientific research departments. I have been told that it was estimated that at least 10,000 man-hours of scientific work was lost due to it, and the suggestion was made that it should be printed in German on leaflets to be scattered over Germany by the R.A.F., so as to cause a corresponding wastage of the war effort over there.

E. V. NEWBERY.

REVIEWS.

Intermediate Dynamics and Statics. By S. L. GREEN. Pp. 298. 12s. 6d. 1951. (University Tutorial Press)

Mr. Green needs no introduction as an author of textbooks on mathematics for he has many to his credit. The present volume was written to meet the needs of students reading for the University of London Intermediate examinations in applied mathematics or for the General Certificate of Education (Advanced and Scholarship Levels) papers in the same subject.

In all, there are twenty chapters in the volume, ranging from the motion of the particle in a straight line to graphical statics. The various chapters on statics and dynamics are mixed and are liberally supplied with examples both worked and set for the reader. Calculus methods are used everywhere, a decision which, I have no doubt, will receive the wholehearted approval of all teachers. Also, the vector notation together with a few of the more essential vector operations are introduced at an early stage in this volume, and are frequently referred to throughout the remaining chapters.

The book is essentially an examinee's textbook written by an experienced examiner with an eye to the modern examination system. The theory certainly covers the bookwork usually set in examinations. The volume ends with over a hundred miscellaneous exercises, an appendix consisting of a few calculus formulae etc., and finally there is a short index to the whole volume.

Apart from the typographical slips noted below the printing is excellent, and the general appearance of the volume attractive.

Errata.

p. 63, lines 14 and 24. In both of the expressions for $\overline{PR}$ and $\overline{PS}$, $\overline{PQ}$ should occur in the numerator and not the denominator as indicated.

J. WILLIAMS.

Starting Astronomy. By E. O. TANCOCK. Pp. 132. 9s. 1951. (Geo. Philip & Son)

The title of this book is distinctly misleading : it is not a book for beginners ; it is much more suitable for sixth-formers.

It may be described as a new edition of the author's " The Elements of Descriptive Astronomy " (now out of print), with additions to bring it up to date. These additions in many cases include more advanced work than was included in the former book.

The new feature of the book is the addition of graphical exercises which are introduced throughout the book ; these are instructive and attractive, but not suitable for beginners. The substance of the former chapter on eclipses has been transferred to the chapters on the Moon, the Sun and Jupiter : eclipses are dealt with more fully than in the former book, but would have been easier to follow if more diagrams had been given. The section on Star Clusters and Nebulae have been expanded into a useful chapter ; and there is an interesting short new section on the Galaxy.

The Questions and Exercises at the end of the former book have disappeared, but many of them have been incorporated in the various chapters. The passages from literature relating to astronomy have been omitted.

The book is scholarly but not easy.

A. W. S.

From Atoms to Stars. By M. DAVIDSON. Third edition, revised and enlarged. Pp. 280, with 56 illustrations. 18s. 1952. (Hutchinson's Scientific & Technical Publications)

This is a substantial revision and enlargement of a book intended to give the elements of modern astronomy to a reader with school-leaving mathematics

and physics. The framework remains that of the first edition, but a considerable amount of attention is now given to some of the more recent theories of the origin of the planets and comets. No doubt there is something to be said for this sort of round-by-round, stop-press account of current research in a book which goes through an edition every 3 years, but it gives a rather breathless air to the last few chapters. Fortunately the reader is warned to have his pinch of salt ready. The rest of the book has been improved by removing the minor blemishes of the earlier editions. A. H.

Initiation à l'Étude du Ciel. By P. PASQUIER. Pp. 66, with 14 figs. and 8 folding plates. 330 fr. 1951. (Paris : Librairie Vuibert)

This little volume is charmingly inscribed to " débutants de tout âge ", and it fulfils its purpose admirably. So long as it continues true that most adults are as unaware of the basic facts of astronomy as children, so long will there be a place for this sort of introduction to the night sky. Not all recent elementary handbooks are as good as this one, however: it may be recommended to all beginners as accurate, simply written and well illustrated with sky charts. Typography, layout, illustrations and paper are all far above the quality much too frequently to be deplored nowadays in French publications, even in serious textbooks.

This book will enable British readers painlessly to brush up their French and their astronomy simultaneously. A. H.

Facts from Figures. By M. J. MORONEY. Pp. viii, 472. 5s. 1951. (Penguin Books)

It is a sign of the growing interest in applied statistics that a book on the subject should be published in the popular Pelican series. Mr. Moroney's book, double the usual Pelican size, is intended for non-mathematical readers. Using only arithmetic and some elementary algebra the author has set himself the ambitious, perhaps too ambitious, task of describing all the commonly used statistical methods, including those relating to Quality Control, Correlation, Analysis of Time Series, and Analysis of Variance. Within his self-imposed limits the author has succeeded admirably in his task, and retains through nearly 500 pages a lively style and a careful attention to detail. The book deserves a popular success.

When reading this book the author's deliberate avoidance of mathematics must be remembered. Mr. Moroney is experienced in both the teaching and the practical application of statistical methods ; he knows that many who would like to understand the principles and methods of statistics are unable to benefit from more mathematical expositions. Is this not a reflection on the effectiveness of the teaching of elementary mathematics, or perhaps on the content of school mathematics courses? Though readers of this review will have a mathematical knowledge beyond that of the readers for whom this book is intended, they will find *Facts from Figures* very useful if they are interested in the teaching of elementary statistics. They will certainly be irritated by the unexplained appearance of strange algebraic formulae and of tabulated values of anonymous functions, but if they are irritated sufficiently to pursue the origins of the formulae and functions in one of the more advanced books recommended in the bibliography, then one of the author's aims will have been achieved.

There are several elementary books which consist of a series of examples which exemplify only the algebraic formulae and the tables in which statistical methods are expressed, leaving the underlying principles unexplained, and sometimes even unmentioned. Readers of such books, having mastered the algebraic techniques, sometimes assume that they have therefore mastered the

statistical principles; they then apply the techniques uncritically, and bring statistics into disrepute. Mr. Moroney, however, is well aware of this danger; as a practical statistician he continually emphasises the practical issues involved in applying the methods he describes and in the interpretation of the results.

Though Mr. Moroney is mainly concerned with the engineering applications of statistics, he is an enthusiast for the application of statistical methods to all fields in which quantitative data are used. Since he is also aware of the limitations of statistics this readable book clearly demonstrates the "statistical attitude" for those to whom the subject is new. Apart from the fully worked illustrative examples in the text, each chapter ends with a few exercises to which answers are provided.

The list of errata at the beginning is unfortunately not quite complete, and the paper covers find it difficult to retain their bulky contents, but these minor defects of the book are more than outweighed by its low cost. Both the author and his publishers are to be congratulated on their enterprise.

B. C. BROOKES.

Committee decisions with complementary valuation. By D. BLACK and R. A. NEWING. Pp. vii, 59. 10s. 6d. 1952. (William Hodge, London)

Under what conditions will a motion and its amendments lead to a majority decision by a committee, each member of which has the various alternatives ranked in an order of preference? The case of three committee members is considered: there cannot be more than one majority decision and a necessary and sufficient condition for a majority decision, if it exists, is that it occurs at a point of maximum preference for at least one member.

The writers then go on to deal with the case where each motion (i.e., original motion and its proposed amendments) has two distinguishable aspects. For instance, one part of the motion might specify the total expenditure to be made on the Navy and Air Force, while the second part specified the amount to be spent on the Navy. In these circumstances a motion is defined for a member by two variables, a and b, and a member's preferences depend on the two variables. There will be for each member a preference function of the form $U = U(a, b)$: with a given valuation of one of the variables—one aspect given—his valuation of the motion will depend on the associated value of the other variable, that is, his valuations will be complementary in nature. A rule is given to show whether a majority decision exists here, by the use of preference surfaces: the necessary and sufficient condition is that any point that is a majority decision must lie on the indifference contours of the three members. The theory is extended to the case where a committee is reaching decisions on two different related topics, each member considering when expressing preferences on one topic that the existing arrangement on the other will continue in operation. In such cases, the writers consider the conditions for "positions of stable equilibrium". The relevant theory in economics (the whole of the work is much influenced by this outlook) is known as the "cobweb theory": in this case the complementary valuation theory can be extended to committees of more than three members. The whole of the treatment is bound up with the assumption that preferences are unidimensional and can be ordered—an assumption that is so often made in other psychological (including pedagogical) work. I do not think this is valid. But the analysis is an interesting extension of some ideas to a problem which has not formerly been the subject of much analysis in this fashion.

It appears from the Prefatory Note that on this account the authors have had some difficulty in getting publication. This is the sort of thing that Karl Pearson found when he first treated biological topics mathematically (the

Royal Society would not accept such work, and Pearson founded *Biometrika* in consequence) and that L. F. Richardson is now finding in his work on war, murder, and peace, with various techniques applied to his treatments of the problems. It is also, no doubt, in part an expression of the difficult times that the learned journals are experiencing with costs rising in all directions. The actual publication of the investigation now reviewed has been made possible " by the generous financial assistance of the Carnegie Trust for Scottish Universities "—one of the authors is on the staff of Glasgow University.

FRANK SANDON.

Contributions to the theory of games. Edited by H. W. KUHN and A. W. TUCKER. Pp. xv, 201. 20s. 1950. *Annals of Mathematics* Studies, 24 (Princeton University Press ; Geoffrey Cumberlege, London)

This Study consists of a series of papers covering several aspects and theorems of both finite and infinite games. The first part of the volume, dealing only with finite games, is restricted mainly to zero-sum two-person games. The Main Theorem, asserting that " A game has a unique minimax value " is first proved in a paper by Weyl and later two algebraic proofs are given. A procedure for effecting a basic solution of the closed convex polyhedron representing a set of optimal strategies is established and a good contribution made towards the problem of the practical computation of the values and optimal mixed strategies for games with a large number of pure strategies ; the solution of a simplified two-person poker is given. Part I ends with two papers on n-person games.

In Part II proof is given that every game has a " value " and generalisations are made to infinite games of the proofs of the Main Theorem for finite games. The last two papers of the Study deal with polynomial games and convex games (that is, games where the pay-off functions are polynomials and convex functions in one of the variables used). Some analogous results to those of the finite games theory and some essential strategies are obtained and a generalisation of a technique for computing solutions of discrete games, which gives a partially constructive way of obtaining these strategies, is given.

The Study is a good attempt to cover the widening range of the Theory of Games and to give an indication of further problems and developments. Although the economic aspects are barely touched except for the short section on infinite games, where some small discussion of coalitions is made, there is here another link with the theoretical Economist.

G. CLARE.

Proceedings of the Second Canadian Mathematical Congress. Pp. xxxi, 259. 45s. 1951. (University of Toronto Press ; Geoffrey Cumberlege, London)

The summer seminar of 1949 and the second quadrennial Canadian Mathematical Congress were held at Vancouver in 1949. Such a well-attended and well-supported gathering, for so it would seem to have been from the descriptions in this volume, makes it difficult to accept Infeld's complaint, that there is insufficient appreciation of scholarship in Canada ; at least, one may doubt whether the fault can be localised.

There are many lectures and short papers printed in the *Proceedings*. One or two struck the reviewer as particularly interesting : Dirac on " Classical and Quantum Mechanics " (" We can construct mathematical theories using any one of these forms, but we don't know which one is the correct one which Nature has adopted, if any ") ; the late A. W. Conway on Hamilton (quoting Sommerfeld, " It seems almost as if Hamilton's methods were expressly created for treating the most important problems of physical mechanics "); and L. Schwartz on " Les mathématiques en France " (" Les livres de Bourbaki ont conquis la jeune génération mathématique en France ").

T. A. A. B.

A School Arithmetic. I. By L. E. LÈFEVRE. Pp. xviii 238. 6s.; with answers, 6s. 6d. 1952. (Black)

The book provides a two-year course for the lower forms of grammar schools, covering the following topics: the four rules for both numbers and compound quantities, fractions (vulgar and decimal), metric system, areas and volumes of rectilineal figures and solids, graphs (introductory), percentages, density and specific gravity, practice, and change of units.

In accordance with his declared intention, the author is at pains to stress, by precept and example, the importance of the pupil's presentation of his solution. In this connection special attention is paid to such points as cancellation, significant figures, the remainder in a division sum, and neat "side work".

The explanations in the text are clear and lively. In such topics as multiplication and division of decimals, and the multiplication of compound quantities, the author is content to give examples of the leading methods (as advocated in the Mathematical Association's *Report on the teaching of arithmetic*), leaving the choice of method to the teacher.

The examples are well graded and numerous. They cover an astonishingly wide range of applications, and there is a good collection of revision papers at the end of the book.

There is a serious misprint on p. 87, and two minor ones on p. 86. A curious method for "inverse bankruptcy" on p. 173 is shown, and as it follows a much better method, it could well be excluded from a future edition. There is an odd and rather pointless digression on "relativity" on p. 133, but the historical notes at the beginning and in chapter 8 are most interesting.

The printing is very clear. Author and publisher are to be congratulated on producing an outstanding book. F. J. T.

Shapes and Sizes. Bk. 3. By J. H. EBBUTT. Pp. 32. 1s. 3d. 1951. (A. & C. Black)

This book has been written for pupils in their last year at the Primary school, but it trespasses on the mathematics of the Secondary school without any great advantage.

The work is well illustrated, clearly set out, and proceeds in easy stages. It deals exclusively with finding the area of various plane figures followed by the surface area and volume of prisms and pyramids. Although experimental work on this subject is within the comprehension of the ten yr. old child most of the formal exercises given are beyond the recommended Primary school syllabus. Over simplification in some cases has led to the printing of mere formulae with little or no explanation or justification. This is particularly regrettable in the sections on the cone and the cylinder, where there appears the misleading statement that:—"The Greek letter called pi, is used to represent $3\frac{1}{7}$", without any reference being made to the all important ratio C/d. In any case the rule for finding the curved surface of the cone is quite out of place at this stage.

Intelligent juniors will profit from following the instructions given for making the nets and models, and by subsequently experimenting with sand for the comparison of volume. It is however suggested that the teacher should exercise discrimination in his choice of the examples. Y. B. G.

Applied Statistics. A Journal of the Royal Statistical Society: Edited by LEONARD H. C. TIPPETT. Vol. 1, No. 1. March 1952. Single number 10s. net. Annual subscription 25s. net. (Oliver and Boyd)

The Royal Statistical Society broke new ground in 1934 by issuing a Supplement to its Journal to cover the proceedings of its newly formed Industrial

and Agricultural Research Section (now the Research Section). More recently the Society's publications have been in two series: Series A (General) and Series B (Methodological). This has not, however, catered fully for the papers read at the various groups of the Society's Industrial Applications Section, nor at the Study Section. The new journal, *Applied Statistics*, is intended to fill the gap, and should be heartily welcomed by all who are interested in the more practical side of statistics, particularly as it applies to industrial processes. In the hands of its Editor, Mr. L. H. C. Tippett, long associated with the British Cotton Industry Research Association, and the Committee which he has gathered together to help him, the material included can be depended on to be fully authoritative; this first number is a particularly good one. The Society is fortunate in the printers (and publishers) it has secured for the new journal, for the type and setting-out are excellent, and a pleasing contrast to the format of the Society's other Journals. J. W.

Tables of the exponential function e^x. 3rd edition. Pp. x, 537. $3.25. 1951. Applied Mathematics series, 14. (National Bureau of Standards, U.S.A.)

This standard table of the exponential first appeared in 1939, opening the series of volumes of the Mathematical Tables Project; a second edition was issued in 1947. The M.T.P. is now the Computation Laboratory of the National Applied Mathematics Laboratories, a section of the National Bureau of Standards, and this volume is now No. 14 of their Applied Mathmatics series.

The contents are

e^x $x=0(0{\cdot}0001)1$, 18 d.; $x=1(0{\cdot}0001)2{\cdot}5$, 15 d.; $x=2{\cdot}5(0{\cdot}001)5$, 15 d.; $x=5(0{\cdot}01)10$, 12 d.

e^{-x} $x=0(0{\cdot}0001)2{\cdot}5$, 18 d.,

e^x, e^{-x} $x=0(0{\cdot}000001)0{\cdot}0001$, 18 d.; $x=1(1)100$, 19 d.; $x=1(1)9\times10^{-k}$, $k=7(1)10$, 18 d.

There is also a new table, from the Ballistics Research Laboratories at the Aberdeen Proving Ground, Md., giving e and $1/e$ to 2556 decimals.

T. A. A. B.

Sur les cercles bitangentes à la parabole. By A. DURAND. Pp. 32. N.p. 1952. (Vuibert, Paris)

This little book contains theorems and constructions on circles having double contact with a parabola. The treatment is mainly by the methods of pure geometry and is, except for a few theorems which deal with reciprocation or three-dimensional space, easy enough to be understood by any student who has learned the elementary properties of the parabola and circle. However, as the spirit of the book is that of Ancient Greece rather than the 20th century world, it can hardly be recommended for a young mathematician.

Teachers may find that many of the theorems make interesting examples.

E. J. F. PRIMROSE.

Dictionary of mathematical sciences. I. German-English. By L. HERLAND. Pp. 235. 24s. 1951. (Hafner Publishing Co.)

This volume gives the British and American equivalents of some 9000 German technical words and phrases, the majority of which belongs to subjects usually considered as mathematical. Despite having to view these as it were from the wrong side, it seems to form a comprehensive and most useful collection. Related subjects, commercial, physical, astronomical and statis-

tical are also well represented, but as only technical words and technical meanings of those words are given, most translation seems to demand the use of an ordinary dictionary as well as the one being reviewed.

Within bounds set by space, important words are treated in considerable detail, and phrases are given to illustrate their use in each relevant subject; these phrases may well provide solutions to some grammatical problems. Perhaps we need more warning of the pitfalls of separable prefixes; and why is "Abstand einer Geraden von einem Punkt" translated as the "distance from a line to a point"? A plentiful supply of cross-references should help in clarifying what shade of meaning is intended, and this dictionary gives many verifications that a compound word is much more than the sum of its parts. Though one may guess these parts from their non-mathematical meanings, one can regret only a loss of picturesqueness in replacing the literal by the correct version of the whole word. The more aesthetic pleasure is, moreover, carried over into the present book in the number of named curves that appear, and the reviewer was led into another visit to Charles Hutton's dictionary to find the whereabouts of a "horopter" (German) = "horopter curve".

The book is clearly produced, and easy to refer to for words that come into its province. It should fill a useful space on one's bookshelf. R. B. H.

Modern Cosmology and the Christian Idea of God. By E. A. MILNE. Pp. 160. 21s. 1952. (Oxford University Press)

The late Professor E. A. Milne infused his own personality into all that he wrote probably more than any other scientific writer in recent times has done. The fact that he evidently wrote this series of lectures as he intended actually to deliver them in speech makes it natural that they should show this characteristic to a particularly marked degree. But they do so even more because they deal with the subjects upon which he had pondered longest and felt most deeply. The result is that in these pages one seems once again to be meeting the man himself as he was in life. At the same time, one knows that he wrote as a man very sick in body, though at the full height of his intellectual powers. The last paragraph of the book opens with the sentence, "My task is concluded". This was penned only a few days before his sudden death. The prescient choice of words strengthens the impression one gets throughout the book that Milne was saying things that he was conscious he might never have an opportunity to say again. For one who knew Milne personally, all this makes it difficult to produce an objective review of his book.

It consists of the ten Edward Cadbury lectures which Milne had been invited to give in the University of Birmingham on the relationship between Cosmology and Christianity today. He died before they could be delivered. The manuscript has been edited and seen through the press by Dr. G. J. Whitrow who can be assured of the gratitude of all readers of the book.

One thing, at least, is easy to say and it concerns much the greater part of the book so far as numbers of pages is concerned. It is that this book provides a comprehensive descriptive account of Milne's kinematic relativity and its applications. He first gave a full account of his early researches in this theory in *Relativity, Gravitation and World Structure* (Oxford 1935) and recently gave a fresh presentation of the whole subject in *Kinematic Relativity* (Oxford 1948). Since the latter book was so admirably reviewed in the *Gazette* (Vol. 33 (1949), 304) by Professor A. G. Walker there is no need here to recapitulate even the main outline of the theory. There are just two items in which the present book goes beyond the earlier ones. Milne took the opportunity to revise his theory of photons in consequence of an inconsistency discovered by A. R. Curtis in the version given in *Kinematic Relativity*; this is the only place in

the present book where the mathematical treatment is presented. Also Milne has included a brief description of a tentative application of kinematic relativity to establish a relationship between gravitation and magnetism: it reproduces some features of a hypothesis advanced in 1946 by P. M. S. Blackett which has attracted considerable notice. The details of this investigation are contained in Milne's last published paper. [A scientific matter discussed here and not elsewhere in Milne's published work, but which is not specifically related to kinematic relativity, is the question of the "heat death" of the universe. He shows the illegitimacy of applying standard thermodynamic arguments to make inferences about the universe as a whole.]

As a general survey of kinematic relativity the book is of great value. This is certainly true in regard to the non-specialist reader whose concern is to learn what the subject is about and to gain a general idea of its procedure. Up till now there has been no such account available, free from technical details. The caveat, that such a reader ought scarcely to expect to acquire material from which to form a highly critical opinion, naturally applies here as it does to a corresponding account of any other scientific development.

The value of the survey is just as great for the specialist reader. It will give him a conspectus of the subject such as he may find it difficult to get from Milne's other accounts with their burden of essential technical detail. Moreover, it will give him the best available record of what Milne himself claimed to be the achievements and significance of his whole enterprise.

This matter of significance leads to the philosophical aspects of the book. The status of *laws of nature* is the main concern of the first two chapters and is recurred to throughout. Stated very briefly, Milne suggests that what we now class as laws of nature will become, or be replaced by, *theorems* in a purely deductive theoretical system. He regards geometry as a science in which this change has already taken place, and he considers that his own work has gone a long way to effecting a similar change in the science of dynamics, as well as in other branches of physics.

Much of what Milne says on this topic is open to fairly obvious criticism if one does not wait to hear him out. For instance, if geometry is treated as a purely abstract subject, as it has to be if it is purely deductive, then there is no necessity for it to apply to the physical universe. We cannot assert that (euclidean) geometry *must* predict, say, the relationships between a set of measurements made on an actual specimen what we call a rigid body. Similarly, if dynamics is rendered purely deductive, we cannot assert that it *must* predict the relationships between observed results in any actual experiment.

Milne did appreciate this circumstance, as is shown explicitly on page 27, for example. But his pursuit of its consequences seems to the reviewer to be scarcely adequate. To begin with, one would have thought that *all* mathematical physics is by its nature purely deductive. The differences between various systems, in particular between Milne's and the rest, seems to be no more than in arrangement and terminology. Even in a theory which appears to be as far as possible removed from Milne's by making constant appeals to experience, the theory itself would be in no wise different in nature, and in particular no more deductive, were the propositions got from appeals to experience replaced by the same propositions restated as additional axioms.

However, the real difficulty comes when Milne assumes the possibility of asserting that "an example of . . . an abstract entity is realized in a particular object in nature". All theories do in fact assume this. Nevertheless, the reviewer holds that it is not possible to make such an assertion, at any rate until it is too late to make any logical use of the information in order to gain new knowledge. The only knowledge of the external world that we can obtain by pure deduction is contained in the information we already possess.

Therefore a deductive theory that claims to predict results of fresh experience, these results being deemed logically necessary, is an impossibility. Unfortunately, it would take too long to give the arguments here but the conclusion is just the old truth that you cannot get more out of a theory than you put into it.

This is all subject to our not discovering some new principle of knowledge (and Milne does not himself claim such a discovery). Also it affords no reason for ceasing to develop physical theories as in the past. And it is, of course, a matter of experience that they do predict the results of experience. All that has been stated above is merely that we cannot *know* that they will do so until the results are experienced. The fact that successful predictions are repeatedly made has been regarded by many thinkers as a matter for wonderment, and rightly so. To a number of them the answer is that " God geometrizes ". Milne's view as developed in this book, is essentially the same. But, because of his belief in the scope of the deductive method, he is lead to state it in such phrases as that " God Himself is limited by reason ".

To leave matters there would be to fail to notice that the title of the book names the *Christian* idea of God. To say, because our experience of the universe corresponds in some way to the results of our mathematical reasoning, that God geometrizes is in itself to say no more than that the universe is what we find it to be. To say that mathematical reasoning is an affair of mind and that we recognize in the universe the activity of a Mind in some way kindred to ourselves is to go further and yet still to be very far from the Christian idea of God. But to mention mind at all is to go outside the accepted domain of physical science. It is in the domains of thought, of the intellectual and spiritual activity of our being, concerning life, mind, personality, mysticism and revelation that we reach the Christian idea. To start with physical science is to start from the most remote domain. Yet it is in a sense to start at the beginning. To start from such fundamental considerations in this domain as Milne does may be the natural way, at any rate intellectually, towards the " larger synthesis " of which he speaks at the end of the book.

W. H. McCrea.

Mathematical logic. By Willard van Orman Quine. Revised edition. Pp. 346. 31s. 6d. 1952. (Harvard University Press; London, Geoffrey Cumberlege)

The number of logicians who have read every page of the three volumes of Russell and Whitehead's *Principia Mathematica* is doubtless larger than popular belief supposes, but the number of mathematicians who have read even a few pages of the *Principia* is probably a very small sample indeed of the mathematical public. Quine's *Mathematical logic* should therefore command a large body of readers for it presents a very concise and attractive account of the field covered by the *Principia* within the covers of a single small volume.

The logical connectives, *and*, *or*, *not*, *if . . . then*, are introduced by Wittgenstein's method of truth tables and are shown to be expressible by means of the single connective *neither-nor*; this is then taken to be the sole connective which forms part of the logical system under construction, the remaining connectives being retained as abbreviations. By comparison with the axiomatic foundation of the propositional calculus, as it is described in Hilbert and Ackermann's *Mathematical logic* (and of course in the *Principia* itself) the truth table method has the advantage of greater simplicity and economy but sacrifices the very elegant theory of independence and completeness which the axiomatic formulation admits.

From the propositional calculus and the definition of tautology we pass straight to a general predicate calculus which makes no restriction on the range of the universal operator (of the kind to which the theory of types has accustomed us). This calculus, which serves to express the notions of " all " and " some ", is set up on an axiomatic foundation, (using axiom schemata, that is, formulae which become axioms when appropriate substitutions are made in them, so that an axiom schema is a summary of an infinite collectoin of axioms). Quine utilises four axiom schemata ; the first permits the introduction of the closure of a tautologous formula as an axiom, the second allows the universal operator to be distributed with implication, the third covers implications such as " ϕ implies $(\alpha)\phi$ " where α is *not* free in ϕ, and the fourth " $(\alpha)\phi$ implies ϕ' " where ϕ' is what results from ϕ by replacing α by α'. These axioms and the rule permitting the derivation of ψ from ϕ and " ϕ *implies* ψ " express the character of the predicate calculus, though no actual formula of the calculus has yet been described.

The next step is to formulate a theory of classes ; this is accomplished by means of a membership connective ϵ which serves to express the idea " x is a member of a class y ". Putting ϵ between variables we obtain the atomic formulae of the predicate calculus. To avoid the reflexive paradox, Quine does not have recourse to a theory of types (and it is here that the chief novelty of the treatment lies), but relies on a distinction between object and element (due to von Neumann), a principle of stratification and an axiom of class membership. Instead of speaking of the class of all objects x such that ϕ, we find " the class of all *membership-eligible* objects x such that ϕ ", and the term " element " is confined to membership-eligible objects, so that " x is an element " is rendered symbolically as $(\exists y)(x \epsilon y)$, where " $(\exists y)$ " is an abbreviation for " $\sim(y)\sim$ ". Thus instead of deriving a paradox from the existence of the class of all classes which are not members of themselves, from the class w such that

$$(x)(x \epsilon w \; . \equiv . \; (\mathsf{E}y)(x \epsilon y) \; . \sim (x \epsilon x))$$

(that is, the class of all *elements* which are not their own members) we derive, taking w itself for x, the harmless conclusion $\sim(\exists y)(w \epsilon y)$ which says just that w is not an element. The principle of stratification says simply that a formula is stratified if and only if a choice of letters can be made for its variables (the same letter for each occurrence of the same variable) in such a way that ϵ comes to be flanked always by consecutive letters. Thus, for instance, the formula

$$(y)(x \epsilon y \text{ implies } (z)(y \epsilon x \; . \equiv . \; x \epsilon y))$$

is stratified. Finally, the axioms of membership are the closures of all matrices of the form

$$\beta_1, \beta_1, \ldots, \beta_n \epsilon \mathbf{V} \text{ implies } \hat{\alpha}\phi \epsilon \mathbf{V}$$

such that ϕ has no free variables but $\alpha, \beta_1, \beta_1, \ldots, \beta_n$ and is formed from a stratified formula by restricting all bound variables to elements ; $\mathbf{V}$ is the class of all elements and $\hat{\alpha}\phi$ is the class of all elements α such that ϕ holds.

In the first edition of the book, Quine omitted the condition, in the membership axioms, that all bound variables be restricted to elements, and in 1942 Rosser showed that a contradiction could be derived in the system. The correction in this revised edition is due to Hao Wang. Of course, there is no certainty that the revised system will not also one day be shown to admit a contradiction, for no consistency proof is known.

The theory of classes is followed by an account of relations and functions. For functional abstraction Church's lambda operator is employed, so that, for instance, $\lambda_x x$ is the identity function, that is, the relation which any element

bears to itself. Formally $\lambda_\alpha\zeta$ is defined to be the relation of any β to α such that β equals ζ.

Following Frege, Quine defines the natural numbers to be classes of classes, with 0 standing for the class whose only member is the null class, and the successor of any number z being the class of classes x which contain a member y such that the remaining elements of x form a class belonging to the class of classes z. Finite classes are defined to be classes belonging to some natural number, and the universal class is proved to be infinite. The arithmetic of natural numbers is set up in detail and then the arithmetic of rational and real numbers is sketched in outline.

In the last chapter Quine sets up a formal language in which to talk about the properties of the predicate calculus. This language contains names "S_1", "S_2", ... , "S_9", for the signs "w", "x", "y", "z", "$'$", "(", ")", "$\downarrow$" and "ϵ" of the calculus and an arch $\frown$ to denote concatenation of expressions, so that for instance the name of the expression "$w'\epsilon x$" is $\widehat{S_1S_9S_9S_2}$. Quine shows that this rudimentary notation, plus the notation of logic, suffices to express the syntax of the predicate calculus. This sort of nomenclature in which the language in which we talk *about* a formal system receives strict formalisation in its turn was initiated by Gödel in 1931, but the formalisation which Gödel used was arithmetical rather than typographical.

The part of the syntax of the predicate calculus which makes no reference to the class membership concept Quine calls protosyntax, and by applying the notation of protosyntax to protosyntax itself Quine proves, by constructing a protosyntactical statement which asserts of itself that it is not a theorem, that no notion of a protosyntactical theorem which is definable by the resources of protosyntax can cover all the truths and exclude all the falsehoods of syntax, and extends the result to show that every protosyntaxtical definition of theorems of the predicate calculus (including class membership) is likewise incomplete. Quine concludes with the observation that protosyntax, though incomplete within itself, is nevertheless syntactically completable, but on the other hand syntax itself is not even syntactically completable.

R. L. GOODSTEIN.

Translations from the Philosophical Writings of Gottlob Frege. By PETER GEACH and MAX BLACK. Pp. 244. 25s. 1952. (Blackwell, Oxford)

Gottlob Frege was outstandingly the foremost thinker in the field of mathematical logic during the last quarter of the 19th century and the first man of the present era to write a book on the foundations of mathematics capable of standing comparison with Euclid's *Elements*. He wrote with a charm and light touch only too rare in German philosophy and yet, despite the brilliance of his mind and the devastating force of his criticism of contemporary thought his work was entirely ignored in his own country. In the preface to his *Grundgesetze der Arithmetik*, Frege, speaking of the causes which had delayed its production for more than ten years, mentions the discouraging neglect of his earlier works. " In vain do we seek ", he writes in a footnote, " a notice of my *Grundlagen der Arithmetik* in the *Jahrbuch über die Fortschritte der Mathematik*. Investigators in the same domain, Dedekind, Otto Stolz, and von Helmholtz, do not seem to know my works. Nor does Kronecker mention them in his essay on the concept of number ". In fact the definition of number which Frege presented in the *Grundlagen* remained unknown until it was rediscovered by Russell in 1903.

Apart from extracts from the *Grundgesetze* (some of which are reproductions of Jourdain and Stachelbroth's translation published in the *Monist* some thirty-five years ago) and a brief passage from the *Begriffsschrift* describing

Frege's symbolic language, the translators have selected some of Frege's most important articles published in various German periodicals. It was in these articles that Frege initiated the logical analysis of language that has proved so fruitful in the past fifty years. With remarkable insight Frege distinguished between the meaning of a word, the object which the word named and the idea which the word evoked in the mind. Failure to appreciate these distinctions has vitiated much that has been, and continues to be, written about logic, mathematics, psychology and philosophy. Frege did not himself undertake the task of analysing these three categories further and this has no doubt led to some misunderstanding of his teaching. By means of these distinctions Frege explains such " mysteries " as " ' 1 + 2 ' and ' 2 + 1 ' are equal and yet not the same ", by observing that ' 1 + 2 ' and ' 2 + 1 ', as symbols, have different senses but refer to the same object (the number three). On the difference between the sense of a sign, the object it refers to and the idea which it arouses in the mind, Frege points to the subjective character of the last by contrast with the objective character of the former. Of course, not every sign names an object, even when we allow the term " object " the very wide connotation that Frege gives it—an object to Frege being anything which is not a function—but it is the central core of Frege's beliefs that numbers *are* objects, not physical or mental objects (like numerals or mental images of numerals) but concepts. Frege defines a cardinal number (as did Russell, later and independently) to be a class of similar classes, but Frege (unlike Russell) distinguishes between a class defined by a membership condition and the aggregate of the members of the class. Thus a null class is the class defined by the membership condition " an object not identical with itself " ; this class has no members so that the attempt to define it by means of the objects which belong to it is bound to fail. The number zero, on Frege's definition, is the concept (class defined by membership condition) " similar to the null class " (" similar " is of course being used in Cantor's sense of (1, 1) related). It is remarkable that Frege did not perceive that the criticism which he levels against a contemporary definition of function in terms of correlation (" how do I tell whether the number 5 is correlated with the number 4 ") applies with equal force against his own definition of number, for how can we tell whether one class is correlated with another or not?

Frege's analysis of the variable in arithmetic has not stood the test of time. If, as Frege believed, the equation $f(x) = g(x)$ between two functions says just that $f(0) = g(0)$, $f(1) = g(1)$, and so on, then no room is left for Gödel's famous function $G(x)$ which is such that each of the equations $G(0) = 0$, $G(1) = 0$, $G(2) = 0$, and so on, holds, but the equation $G(x) = 0$, with a variable x, is *not* provable in a suitably formalised number theory.

Frege is at his best in his overwhelming attack on the Formalists. " If numbers are taken to be tangible figures, whose existence is rendered certain by their tangibility, why then they must be subject to all the limitations of such a material existence." Thus, if a sequence of numbers $a_1, a_2, a_3, \ldots$ is defined to be infinite if no term in it is the last, so that in accordance with a given prescription new terms can always be constructed, then all sequences become infinite, or all finite, depending upon the sense we attach to " can' ". If we presuppose human ability then no sequence will be endless ; if we are satisfied just with *possibility* then any sequence *can* be continued and is therefore endless. An infinite sequence of figures would consist " first of terms which are written down, secondly of terms which are not written down but can be written down, and thirdly (so it appears) of terms which cannot be written down ". But the argument must be read in its entirety, for it is the wealth of accumulated evidence that is so impressive, not just the skill of individual thrusts.

The translation concludes with the Appendix to the *Grundgesetze* on Russell's paradox. Here Frege tells the story of the contradiction which Russell discovered in the foundations of the theory of classes and communicated to Frege as the second volume of the *Grundgesetze* was passing through the press. As the translators stress in their preface, Frege wasted no time in bemoaning the blow which had befallen him, but started at once to try to work out a solution of the paradox. Black likens the solution outlined in the Appendix to that which was later followed by Quine in his *Mathematical logic*, but the resemblance is by no means obvious.

These selections form the second book of translations of Frege's work which Basil Blackwell has published. The first was Frege's *Grundlagen*, admirably translated by J. L. Austin. Students of mathematical logic owe a great debt to the publisher for his courageous venture.

R. L. GOODSTEIN.

Linear differential equations, ordinary and partial. I. Ordinary equations. By E. W. TITT. Pp. vii, 222, xi. $3. 1951. (Mathematics, Research, and Publishing Co., Austin, Texas)

The method and scope of this book are indicated clearly in the preface, where the author writes: "Part I represents an attempt to turn Lagrange's idea of the adjoint and Green's idea of a suitably specified solution of the adjoint into a practical method of solving linear differential equations. The solution of a linear equation with constant coefficients is divided into three steps. The first step involves the determination of the roots of the characteristic equation and in the case of a system of equations this step also involves the determination of a set of relative amplitudes corresponding to each root. The second step involves the determination of an integrating factor, or a set of integrating factors, as the case may be. This is accomplished by the solution of a system of linear algebraic equations, the equations of this system being obtained from a set of formulae varying with the nature of the roots. Step three is the assembly of the solution with the assistance of integration tables covering the usual type of forcing terms. This assembly is carried out in tabular form.... In the case of simultaneous equations the handling of the theory is facilitated by the use of tensor notation.... In Chapter IV we apply the ideas of Lagrange, Green and Hilbert to an equation with variable coefficients in the study of Bessel functions. These ideas are used to set up integral equations characterizing the various Bessel functions."

It is clear that the present work (Part I) must be judged on its performance in relation to the method and scope of the work outlined above, and this is concerned mainly with Chapter II, "The ordinary linear differential equation", and Chapter III, "Systems of linear equations with constant coefficients". It is difficult to avoid the observation that Chapter I, "Ordinary differential equations of the first order", has little or no connection with the rest of the book. It is probably intended as an introduction or may have been added as an attempt to make a complete treatise from what is fundamentally one special method. This chapter gives most of the ordinary methods for solution of first-order equations, has some illustrations of the method of sketching solutions (*cf.* Piaggio, pp. 5–9, or Chapter I of Levy and Baggott, *Numerical studies in differential equations*), and also gives some interesting examples of singular solutions, though there is no attempt to give a complete treatment of the latter. At the end of this section we read (p. 39): "Thus it seems to us that physics must be the guide in determining what we mean by the solution of a differential equation," and again: "The final test is whether the solution thus obtained seems to agree with physics." If this refers to the

preceding section, the reference should be to geometry, rather than to physics for the work to which it apparently refers is the relation of the curves representing singular solutions to those which represent ordinary solutions of the differential equations, but much work in other parts of the book deals with physical applications, especially electrical and mechanical ones.

With regard to the main part of the work, Chapters II and III, the author gives a definite method, or sequence of methods, using the fundamental principle outlined in the preface, which solves completely all the equations given and, in each case, satisfies the given initial conditions. Thus, for the comparatively simple case of a linear differential equation of the second order with constant coefficients,

$$Ay_{tt} + By_t + Cy = f(t), \quad (y_t = dy/dt)$$

u is a solution of the adjoint equation

$$-Au_{tt} + Bu_t - Cu = 0$$

such that $u(\tau) = 0$, $-Au_t(\tau) = 1$. Then

$$y(\tau) = \Big[-Ayu_t + (Ay_t + By)u \Big]_{t=a} + \int_a^\tau uf\,dt.$$

$y = e^{\gamma t}$, where $A\gamma^2 + B\gamma + C = 0$, and

$$-Au_t + (A\gamma + B)u = e^{\gamma(\tau - t)}.$$

There are modifications of the last equation given by different formulae according as the two values of γ are real and different, real and equal, or complex. There are more cases for equations of the third and fourth orders, and the evaluation of the integral $\int_a^\tau uf\,dt$ requires the use of a formidable table of integrals given in Appendix I, while Appendix II gives a summary of the formulae used throughout the book. The methods given cover all the usual cases for which most books use either the operator D or the Laplace transform, that is, the cases where $f(t)$ involves $\sin at$, $\cos at$, e^{kt} and polynomials in t, and is also completely satisfactory for examples of the types (p. 64):

(17) $\quad y_{tt} - 2y_t + 10y = e^t \sec 3t, \qquad y(0) = 1, \qquad y_t(0) = -1;$

(18) $\quad y_{tt} + 3y_t + 2y = \sqrt{(1 + e^t)}, \qquad y(0) = y_t(0) = 1.$

Titt is concerned almost completely with the practical solution of differential equations by a uniform method, choosing the appropriate formulae and the table of integrals as necessary. Comparison with the two methods mentioned above (operator D and Laplace transform) is inevitable. In most cases tried, the operator method with arbitrary constants followed by elementary algebra to satisfy the initial conditions was found to be quicker, but of course, that might be altered by continuous practice with Titt's method. It will be noted that the operator method must be supplemented by another method for examples such as (17), (18) above. Thus (17) is solved quickly by the substitution $y = ze^t$ and the method of variation of parameters for the equation for z. It will be noted that Titt's method is not the ordinary method of Green's functions (*cf.* Ince, *Differential equations*, p. 254, and Titt's work on pp. 117–119), depending on the given differential equation, as Titt's method uses the adjoint equation, but there is the close connection given by Green's formula,

$$\int_a^b \{vL(u) - uL(v)\}dx = \Big[P(u, v) \Big]_a^b,$$

which follows the Lagrange identity, though Bateman (*Differential equations*, p. 276) says that the formula is due to Abel. It may also be considered that

Titt's method is similar to that of the variation of parameters, especially in the conditions imposed on u.

The work in Chapter IV, " Bessel functions ", where the coefficients in the differential equations are no longer constant, becomes complicated, and it is difficult to see any advantage here over the usual method of solution in series or the definition of $J_n(z)$ as the coefficient of z^n in the expansion of

$$\exp\{\tfrac{1}{2}z(t-1/t)\}$$

for the first kind of Bessel function.

The present work gives the titles of Chapters V–XII, mainly on partial differential equations, forming Part II to be published at an early date, and, to quote again from the preface, " The present work on ordinary equations was undertaken in order that the student would not have to take on an entirely new set of ideas in the transition from ordinary to partial differential equations ".

Very few errors have been found, though, as indicated above, there are signs that Chapter I was written with less care than the rest of the book. There is a superfluous factor (x^2y^2+1) in the answer to Ex. 4, p. 5, while in the twofold answer to Ex. 6, p. 31,

$$\text{(i)}\quad x^2-y^2=\exp\{2\,\text{artanh}\,(y/x)+c\},$$
$$\text{(ii)}\quad -x^2+y^2=\exp\{2\,\text{arcoth}\,(y/x)+c\},$$

the 2 in each index in the right-hand side should be omitted, and both results can be replaced by the simple form

$$(x-y)^3(x+y)=c.$$

In Ex. 6, p. 40, the equation

$$yy_x{}^2-xy_x{}^2+xy_x-x^2=0$$

should read, with the usual notation,

$$(y-x^2)p^2+xp-x^2=0.$$

In Ex. 4, p. 20, the term $5xy^3$ should be $10xy^3$. F. UNDERWOOD.

Arithmetical Questions on Algebraic Varieties. By B. SEGRE. Pp. x, 55. 10s. 6d. 1951. (University of London, the Athlone Press)

The investigation of both the arithmetic and the algebraic-geometric properties of manifolds defined by one or more polynomial equations in the variables is interesting, important and often very difficult. It is evidently well worth while to give even a brief account of the interaction of these two points of view as is done in the present book, and the more so since the coefficients of the polynomials may be given as elements of a field different from of the usual rational or complex number field.

Among the arithmetical questions is that of finding the points on a manifold lying in some assigned domain of rationality, for example, the rational number field. When the manifold is one-dimensional and represents, say, a curve in two-dimensional space, Poincare showed some fifty years ago that great progress could be made by considering the algebraic-geometric aspect and properties of the manifold. He was in the fortunate position of being both a geometer and an arithmetician and so had at his disposal the two corresponding techniques.

For two-dimensional manifolds, however, there was for many years no one who combined both geometrical and arithmetical interest and knowledge. In the early 1940's, at a time when I was specially interested in the question of the rational points on cubic surfaces, Professor Segre, the distinguished Italian geometer, joined me at Manchester. He soon became interested in the prob-

lem and the fusion of geometrical and arithmetical interests soon took place. The result was that since that time Professor Segre has made and published many contributions on this and related topics, in fact, some twenty-four papers since 1943. A sketch or outline of some of these is contained in his book, which consists of three lectures delivered in 1950 in the University of London. The large bibliography at the end of the book includes a total of some seventy-eight papers and testifies to the width and interest of the questions involved in his lectures.

The first chapter deals with the geometry of quadratic forms in arbitrary commutative fields; the second with algebraic geometry and diophantine analysis; and the third with problems concerning rationality and the theory of the base. In chapter II, the results on the rational points on a cubic surface S and the representation of S by planes in the rational field by 1–1 correspondence are particularly interesting as are also the arithmetical properties of cubic primals. The third chapter includes, *inter alia*, the extensions of the Bring and Jerrard theorems on the Tschirnhausen transformation reducing the quintic equation to its simplest normal form. Though not relevant to Professor Segre's lectures, it might be mentioned that this normal form is fundamental in Hermite's beautiful solution by elliptic functions of the quintic.

This excellent little book is interesting and suggestive and well worth while reading, not only for the expert but also for anyone who wishes to get a bird's eye view of some fascinating parts of pure mathematics and who is more concerned with ideas than with detailed proofs. L. J. MORDELL.

Introduction to Modern Prime Number Theory. By T. ESTERMANN. Pp. x, 75. 12s. 6d. 1952. Cambridge Tracts in Mathematics and Mathematical Physics, 41. (Cambridge University Press)

The author's " main purpose in writing this book was to enable those mathematicians who are not specialists in the theory of numbers to learn some of its non-elementary results and methods without too great an effort ". He assumes nothing in the theory of numbers that is not given in Hardy and Wright's *Introduction* and proves three main results. The first is the Prime Number Theorem in the form

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(xe^{-c\sqrt{\log x}})$$

for a suitable positive c. The second is the corresponding result for primes in an arithmetic progression $kn+l$, holding uniformly in k for all $k \leqslant \log^u x$ and any fixed $u > 0$. The third is Vinogradoff's famous theorem that every sufficiently large odd positive integer can be represented as a sum of three primes.

The exposition is masterly in its clarity, ease and economy. For example, the proof of Vinogradoff's theorem takes only 16 pages, which includes a careful explanation of the main ideas before the detailed proof begins. Until I read this book I would not have believed it possible to give so lucid and simple an account of the proofs of these three difficult and important theorems. The author has put specialists as well as non-specialists in his debt. E. M. W.

A new calculus. III. By A. W. SIDDONS, K. S. SNELL and J. B. MORGAN. Pp. vi, 463. 25s. 1952. (Cambridge University Press)

This final volume of *A new calculus* (the first two parts of which were reviewed in Vol. XXXVI, No. 315, p. 62) provides an ample course for Scholarship candidates; and by incorporating the chapters headed Double and Triple Integration, Fourier Series, Introduction to Analysis, and Differential Equations (in which the treatment is similar to that given in Piaggio),

the authors have ensured that their book will form a useful introduction to University courses.

One of the hardest tasks of the schoolmaster in teaching pupils who will later read mathematics at the University consists not of showing them how to solve examination problems, but of answering the question familiar enough to readers of the *Gazette*: "Why all this fuss?" Differentiation under the integral sign, integration of infinite series, conical projection, to mention only three familiar processes, can sometimes be used to obtain quick solutions. But are the processes justifiable? The teacher of Scholarship candidates is under the obligation of encouraging his pupils to see that such questions must be asked, and answered; and his task is not made simpler by the fact that there is often little time available for teaching such pupils, who must therefore learn largely by reading the text-books recommended to them.

For this reason, the reviewer hopes that this volume will be widely recommended. It is a sound, well-written text-book which students can safely be left to read for themselves. The worked examples are aptly chosen, and there are excellent sets of exercises (from various sources) for the student to work. No one using this book need fear that he will learn anything that will later have to be unlearnt.

Misprints are few; the only misleading one discovered appears in a note on p. 20, where the definition of $\sin\theta$ is incorrect.

It need hardly be added that the book is beautifully set out and printed.

R. WALKER.

Graduate study in education. Fiftieth Yearbook of the National Society for the Study of Education, Part I. Pp. xix, 369. 21s.; cloth, 26s. 1951. (University of Chicago Press; Cambridge University Press)

Post-graduate work is an important and highly-organised part of the American educational system, and so is worthily granted a place in the series of yearbooks of the National Society for the Study of Education. But even a large volume such as this would be hopelessly inadequate to deal in any detail with the whole field, and therefore the present book deals only with the comparatively small part of post-graduate study, that in which the topic of study is education; more precisely, that branch of post-graduate work which is "designed to serve the intellectual interests and professional needs of those whose present or prospective careers are identified with the field of organized education". The first 130 pages deal with the history, aims and general organisation; the remainder of the volume offers examples of the programmes provided by some of the institutions which cater for graduate study in education. The first part of the volume is likely to be more interesting to the average English reader than the latter part; the close analysis and the detail, at times minute, may become tiresome, but the undercurrent of idealism must command respect, even in England, where education has seldom been taken seriously.*

T. A. A. B.

Heat. By R. C. BROWN. Pp. ix, 277–547, viii. 13s. 6d. 1952. (Longmans Green)

This is Volume 2 of a Textbook of Physics for Intermediate and General Certificate of Education (Advanced Level) students. It maintains the high standard set in the first volume.

J. TOPPING

The Calculus. By G. P. RAWLINGS. Pp. 84. 10s. 6d. 1951. (Percival Marshall)

This is a short book, designed to dispel fear of the calculus by a simple

* Scotland, I understand, must be excluded from this judgment.

treatment. It assumes a knowledge of ordinary-level G.C.E. mathematics, and is addressed both to the general reader and the student.

Part I (20 pp.) explains that calculus employs two main operations, integration and differentiation, and outlines the kind of problem for which it is useful. Given a curve, integration corresponds to the construction of a curve the ordinate of which at any point is a measure of the area under the given curve up to a corresponding point, and differentiation to that of a curve which gives the gradient of the original curve. Part II (12 pp.) describes how these "integral" and "derived" curves can be constructed, and demonstrates graphically that integration and differentiation are inverse operations. It is by this graphical approach that the author hopes to make calculus simple. It should be effective, but would be more so if the reader were given a few easy examples to work through. He will need confidence before tackling Part III.

This begins easily enough. The slope of $y=mx+c$ is proved to be m, is defined to be the d.c. of y, and is written dy/dx. The slope of $y=x^2$ at $x=p$ is then proved by limits to be $2p$; so, if $y=x^2$, $dy/dx=2x$. The d.c.'s of x^3, x^4, ... x^n are covered by "similarly", and these functions can now be integrated. The notation δx, δy is not used. Graphical methods are employed to differentiate $\sin x$, $\cos x$, and e^x, but the need for radian measure in the two former has to be assumed. The rules for differentiating sums, products, quotients, and functions of a function are then stated without proof, and used for finding the d.c.'s of the remaining trigonometrical functions, direct and inverse, and the logarithmic function. A table of standard forms follows, in which d.c.'s are unfortunately called "differentials". This Part III of 20 pp. is a formidable assignment. The beginner needs gentle coaxing, with easy examples every few pages to try his hand at. He must have time for what is novel to become familiar.

Part IV (28 pp.) consists entirely of applications, such as rates of change, maxima and minima, errors, volumes, work, centres of gravity and pressure, moments of inertia, and S.H.M. In some of these the integral as a limiting sum creeps in without explanation. That is a pity, The method is of fundamental importance, and deserves some justification, however slight. The author is consistent in that Part IV uses all the technique established in Part III, but these two parts cover so much ground that the reader may be satiated before the end, rather than eager for more. A. H. G. PALMER.

A Philosophical Essay on Probabilities. By PIERRE SIMON, MARQUIS DE LAPLACE. Translated from the Sixth French Edition by F. W. Truscott and F. L. Emory. Pp. viii, 196. $1.25 (Paper); $2.50 (Cloth). 1951. (Dover Publications, New York)

This book is a new edition of a translation first published in 1902 (though no indication of this fact is given by the present publishers). The only new feature of the new edition is an introduction by Professor E. T. Bell which briefly explains the scope and purpose of the Essay. Though the Essay was first published separately as a popular work addressed to the non-mathematical but well-educated reader, it was later incorporated as an introduction to the third edition of Laplace's great work *Théorie Analytique des Probabilités* (1820).

The purpose of the Essay was to show how far-reaching are the applications of probability theory in everyday life, and, to quote from its conclusion, "that there is no science more worthy of our meditations and that no more useful one could be incorporated in the system of public instruction". To achieve this purpose Laplace first describes the general principles and the analytical methods of the theory; he then illustrates their application to many human activities, including games of chance, the moral sciences, "decisions of assemblies", and "judgments of tribunals".

The most difficult part of the task is to explain the analytical methods of the theory to non-mathematicians. Laplace's method is simply to describe his analysis in words instead of symbols. The method fails because, until the words are translated back into symbols, the text is unintelligible even to a mathematician. A quotation may illustrate the point: "Indeed the nth difference of the primitive function having for a discriminant function the product of V by the nth power of the binomial one divided by t less unity, the primitive function which is the nth integral of this difference has for a discriminant function that of the same difference multiplied by the nth power taken less than the binomial one divided by t minus one, a power to which the same power of the character Δ corresponds; this power indicates then an integral of the same order, the index x varying by unity; and the negative powers of δ indicate equally the integrals x varying by i units. We see, thus, in the clearest and simplest manner . . ." (p. 41). It is difficult to disagree with Todhunter, who condemns the analytical chapters as "a complete waste of space".

Fortunately the analysis forms only a small part of the book. If the reader skips the theory and turns to the applications he will be rewarded by finding some entertaining discussions of all kinds of problems in which probability plays a part. Unhappily, this part of the book is marred by defects of translation; the translation is so precisely literal that the reader who knows French will find himself translating the text back into its original language, and the reader who does not know French will be irritated by the unfamiliar style.

A new edition of a work of historic importance and interest is always welcome, though Laplace's great contribution to the development of probability theory deserves worthier expression than it is given here. Nevertheless, in spite of its two serious defects, this book should be widely read, both for the insight it gives into the mind of a great mathematician and for its interesting discussions of some classical problems. B. C. BROOKES.

L'enseignement des mathématiques générales par les problèmes. I. By G. BOULIGAND and J. RIVAUD. Pp. vi, 372. 2000 fr. 1951. (Vuibert, Paris)

Because time is lacking, many University lecturers find it necessary to confine themselves to a development of the theoretical foundations of their subjects, and are unable to digress sufficiently to illustrate its applications to the solution of problems. This first volume of a two-volumed work recognizes the fact and aims at providing assistance to the student who is left largely to fend for himself outside the lecture-room and who, with "finals" examinations ahead, requires further guidance.

The chief topics dealt with are vectors, theory of equations and determinants, and calculus (with its applications to curves in two and three dimensions), and the twelve chapters included in this volume consist largely of worked examples, with appropriate references to sources in which basic theory can be read.

The book is lucidly written, and University students in their first and second years would find it most useful. It would also be a valuable addition to the mathematical shelves of a school library, for it contains much (particularly on the applications of vectors) that the abler pupils could profitably read, besides providing them with necessary practice in the reading of mathematics in a foreign language. R. WALKER.

Einführung in die freie Geometrie ebener Kurven. By L. LOCHER-ERNST. Pp. 85. Sw. fr. 12.50. 1952. (Birkhäuser, Basel)

This is the first book of a new series—Elemente der Mathematik vom höheren Standpunkt aus—not to be confused with the series with a similar title by Felix Klein. The aim is to present accounts of various branches of

mathematics which may be understood by the non-specialist ; for this reason, as little previous knowledge as possible is assumed.

This treatment of plane curves skilfully combines the rigour of the axiomatic approach with the informality of the intuitive approach. It starts from an unfamiliar set of axioms, of which we give a short outline. After postulating the usual axiom that two points determine a line (and its dual) the author introduces as undefined terms a motion of a point along a line (and, dually, the motion of a line about a point) and the order in which the points of a line appear in this motion : an axiom is then introduced which ensures that there are two possible motions on a given line. The reason for choosing these particular undefined terms is that they correspond to one's intuitive ideas of how a line may be described. Finally, there is a continuity axiom similar to Dedekind's.

The word " freie " in the title corresponds roughly to the English " pure ", that is, no coordinate system is used ; in fact, the axioms do not permit a coordinate system to be set up. A great gain is that no restriction to algebraic or even to analytic curves has to be made.

Curves are introduced by means of *elementary arcs*, which are defined by means of properties all of which are intuitively acceptable. A closed elementary arc is called an elementary curve. The author then defines a *simple arc* and a *C-arc*, which is so-called because of its resemblance to the letter *C* (the tangent at one end-point passes through the other end-point). A closed *C*-arc is called an *Eilinie*, which we should call an oval.

The degree of a curve is defined as the maximum number of points in which a line meets it ; it follows that an oval is of degree 2. A complete treatment of curves of degree 3 is given. Finally, it is shown how to form a new curve from a given one by " unravelling " double points and double tangents.

At the end of the book are a collection of examples and a description of two unsolved problems. One of these concerns the relations between the degree, class, and the numbers of cusps, inflexional tangents, double points and double tangents (in other words, the Plücker equations for this type of geometry), and the other is the number of essentially different ways in which the projective plane may be divided by n lines, no three of which are concurrent.

Mathematicians will probably find the book interesting, because of the unusual approach to the subject and because the thoroughness with which it is treated does not detract from its beauty. It is difficult to forecast what the opinion of non-mathematicians will be : the author hopes that artists, among others, may like the book because of the insight it gives into geometrical form. If they are not familiar with the axiomatic development of a subject, they may find it difficult, though it is pointed out that for an understanding of the sections on curves only the results, and not the proofs, of the first section need be studied.

If inexperienced students read the book they should be warned that some of the definitions are different from those usual in algebraic geometry. For example, if P is a non-multiple point of a curve, and p is the tangent at P, the number of intersections of p with the curve at P is counted as either 2 or 3, depending on the type of tangent, whereas in algebraic geometry the number of intersections can be any integer greater than 1.

The book is well printed and there are 168 excellent diagrams. These are collected together at the end, which is no doubt convenient for the printer, but rather distracting for the reader. It is unfortunate that the book is rather expensive (over £1 at the present rate of exchange).

The idea of the new series is excellent, and in this first volume Professor Locher-Ernst has set a high standard, which we hope will be maintained.

E. J. F. P.

Theorie und Anwendung der Funktionen einer komplexen Veränderlichen. By J. HEINHOLD. Pp. 213. DM. 15. 1949. (Oldenbourg Verlag, Munich)

The book aims to provide an introduction to functions of a complex variable for students of science and technology. It starts at the beginning with the definition of a complex number, and ends with Cauchy's integral theorem, the maximum and minimum theorem, Liouville's theorem, Mittag-Leffler's theorem on partial fractions, and the use of Poisson's integral in the solution of the boundary-value problem for the circle.

The foundations are briefly but carefully laid, including the simplest properties of sets of points and the convergence of series and products, assuming only an elementary knowledge of differential and integral calculus and the elements of analytical geometry. Applications are mainly confined to conformal representation by means of the elementary functions, and the corresponding two-dimensional problems of potential theory. The emphasis is rather on the mathematical than on the practical side; thus, while the Joukowski profile is mentioned, there appears to be no discussion of the Schwarz-Christoffel transformation such as might be expected in a book of more than 200 pages intended for students interested in the problems to which the theory can be applied.

The book is illustrated with portraits of Gauss, Cauchy, Riemann and Weierstrass. There are a dozen sets of examples, which are worked out in an appendix of 22 pages. F. B.

Algebraic Projective Geometry. By J. G. SEMPLE and G. T. KNEEBONE. Pp. viii, 404. 35s. 1952. (Oxford University Press)

It is not so long since it was usual, when lecturing on geometry to mathematical honours students, to apologise for the lack of an adequate text-book. The last few years have seen a great change: the difficulty today is rather one of selecting from the steadily growing number of books which have been written more or less expressly to fill an obvious gap in the literature. The present volume, none the less, will receive a warm welcome, since it provides an account of all the topics which should normally find place in an honours course, presented with a simplicity and clarity of treatment which should make it easy to use in a class of students of varied attainments. At the same time, it contains hints and suggestions which should attract the attention of future specialists.

After two well-written informal chapters which set out to link the ideas underlying projective geometry with those of the elementary geometry with which the reader will already be familiar, the authors proceed to the formal development of their subject. In accordance with modern trends, the foundations are algebraic rather than synthetic in character; but, while algebraic methods are used freely, the authors always make it clear that they are discussing geometry, and the spirit of the book is geometrical throughout. After introducing the fundamental ideas of projectivity and cross-ratio the book deals with the usual properties of conics, quadrics, twisted cubics and line systems, with a good treatment of elementary correspondence theory. A notable feature is the attention given to collineations in the plane and space, though the general theory of invariant factors is not touched upon. Some emphasis is placed on affine and metrical specialisations of projective theorems, and the examples selected for this purpose are interesting and at times unusual. A very interesting final chapter deals briefly with the possibilities of geometry in higher space, which should serve to whet the appetite of the attentive reader.

In a subject which abounds with pitfalls, the authors have maintained a high standard of rigour. A few oversights, however, may be noted for attention

when a revised edition is called for. The most serious of these is the omission of a proof that a line is determined, and determined uniquely, when a set of Plücker coordinates satisfying the identical relation is given. The proof of the converse of Pascal's theorem is a little too sweeping, since unless the statement of this theorem is suitably restricted some awkward special cases require consideration. Finally, the proof of Poncelet's porism contains a curious flaw, logical rather than geometrical, which needs a change of wording to make the argument convincing. As this proof is one commonly given in text-books it is perhaps worth while making the point explicit. We are given two conics, S, S', and an n-gon $A_0A_1 \ldots A_{n-1}$ inscribed in S whose sides touch S', and it is required to show that there is an infinity of such n-gons. Starting with a general point P_0 of S we construct points $P_1, P_2, \ldots P_n, \ldots$ on S such that $P_{i-1}P_i$ touches $S'(i=1, 2, \ldots)$. The argument now proceeds as follows. (i) P_0, P_n correspond in a symmetric (2–2) correspondence on S, hence (ii) P_0P_n envelops a conic Σ, (iii) Σ touches S at $A_0, \ldots, A_{n-1}$, since these are double coincidence points of the correspondence, and hence coincides with S, whence (iv) $P_n = P_0$ for the general point P_0, and the theorem follows. The difficulty is that the deduction (ii) is only valid, in general, for irreducible correspondences, while $P_n = P_0$ implies that the correspondence is reducible (it is actually the square of the identical correspondence). The necessary modification in the proof is clear if one realises that every symmetric (2–2) correspondence which is reducible necessarily contains identity as a component.

The authors' style is always clear and readable, there is a useful collection of exercises and an index. One curious observation which the reviewer failed to comprehend is the statement on p. 190 that Serret's theorem has, strictly speaking, nothing to do with apolarity ; it seems to me that, on the contrary, it has very little to do with anything else.

The imprint of the Oxford Press is sufficient guarantee that the printing and lay-out of the book are of a high standard. This interesting book may be warmly recommended.

J. A. TODD.

The Theory of Relativity. By C. MØLLER. Pp. xii, 386. 35s. 1952. (Oxford University Press)

Suppose some one has no idea of what a cathedral is. If you want to enlighten him, you will try to show him one before you start talking about what sort of foundations a cathedral should have or what you consider to be the best way of building a cathedral. Now the theory of relativity, or at any rate that of general relativity, is a beautiful and impressive structure. But anyone who has not seen the structure as a whole cannot know that it is a subject the foundations of which are worth studying and the erecting of which produces numerous interesting technical problems.

Any student who has worked through this book will be well-equipped for what may be called the " engineering " work of relativity theory, as regards both the laying of foundations and the setting up of the structure. But, so far as the reviewer can judge, he will have no adequate conception of what the structure itself is intended to look like. An explanation is perhaps to be found in the opening phrases of the preface. The author says that " The present monograph . . . is primarily a textbook. . . ." This is a contradiction in terms, if the terms " monograph " and " textbook " are intended to have their customary meanings. It seems to the reviewer that this confusion of purpose persists throughout the book.

As a textbook, it will give the student no sense of form in the subject as a whole. But he can learn a lot about important sections of the subject provided he takes on trust the fact that they are important. As a monograph, upon many points it will enlighten the reader who already knows something of the subject.

But such a reader will not need to be presented here with the rudiments of tensor calculus and differential geometry. In passing, one must remark that tensor calculus and differential geometry have not been ill-expounded over the past thirty years, and, moreover, that several first-class books on them have appeared quite recently. It is therefore high time finally to abandon the traditional assumption that the student of relativity theory knows nothing about them or cannot read the textbooks if he has to.

Professor Møller's book deals with " classical " relativity theory, that is, he excludes the discussion of unified field theories. He takes the subject as far as a discussion of the Einstein and of the de Sitter universe, but excludes modern developments in cosmology. The topics included are in the main the standard ones within this scope, and it will suffice here to indicate briefly some of the special features of their treatment.

Professor Møller says more than is usual about the most general Lorentz transformation. This is very helpful, and it enables him at the appropriate stage to treat such matters as the " Thomas precession " which are not given in many books. His discussions of wave-propagation and of the theory of continuous media are particularly full and illuminating. He gives the relativity forms of certain basic variational principles. He goes further than usual in showing the extent to which general relativity conforms to " Mach's principle " ; however, he does not express his treatment in such terms. His use of a rotating frame of reference in illustrating his arguments about the foundation of the general theory brings out many important points, though his concept of " standard measuring-rods " and " standard clocks " seems to require further justification. Throughout the book he makes a special point of preserving the distinction between space and time and of not conceding everything to a four-dimensional treatment. Well-considered as are his reasons for this, it does contribute to hinder his giving the work any well-defined shape as a whole.

Something must be said about the references. Professor Møller gives a large number of very useful ones ; his author-index includes over one hundred names. However, amongst others, one misses any mention of Eddington, Eisenhart, Larmor, Mach, McConnell, Milne, Riemann, Schrödinger, Whitehead and Whittaker, parts of whose work comes within the scope of the book. In such a large subject, an author has a considerable choice of topics and treatment ; clearly he is not going to be able to mention all the previous contributors within his general scope. But to contrive to write nearly 400 pages on the particular topics he does choose and to give a couple of hundred references, without mentioning *any* of the names just listed is surprising, to say the least. Incidentally, one cannot help noticing how Professor Møller has taken the liberty of making omissions, additions or alterations in the initials of not less than half-a-dozen well-known authors in his own list.

W. H. McCrea.

Elements of the Topology of Plane Sets of Points. By M. H. A. Newman. Pp. viii, 214. 27s. 6d. Second Edition, 1951. (Cambridge University Press)

The first edition of this book, which appeared in 1939, did us the great service of providing an introduction to topology that the reader could work through in a systematic manner, acquiring new ideas one by one, and testing his grasp of them by attempting the numerous problems furnished for this purpose. We now welcome the appearance of a second edition, embodying revisions both " in the large " and " in the small ".

In Chapter I (the theory of sets) the main change has been the adoption of the current " cup " and " cap " notation for set union and intersection. In conformity with recent trends, the emphasis in Chapter II (metric spaces) has been transferred from limit-points to open sets ; the treatment of vector

spaces is more general, and a new section on complete metric spaces has been added. In the next chapter " homeomorphism " now receives its conventional (though etymologically inaccurate) spelling ; there is less use of convergence, and more of open and closed sets, in the discussion of homeomorphisms and continuous mappings. In Chapter IV the section on locally connected sets has received extensive revision.

In the second, more combinatorial, portion of the book the changes are more sweeping. The arrangement of the chapters has been altered so as to distinguish clearly between homology and homotopy properties. The homology groups are introduced explicitly, and the basic notions of homotopy theory are carefully explained. In a new section on the intersection and orientation of paths in the plane the author discusses the relation between homology and homotopy for simple closed curves, and the notion of the degree of a mapping. Besides the applications of topological arguments to Cauchy's theorem, most of which appeared in the first edition, we now find a brief discussion of Jacobians and implicit functions. One change of a technical nature is the distinction now made between a chain (" complex " in the first edition) and the set of points that is its locus.

To make room for this new material the sections on boundary elements of domains and on the connectivity of closed sets have been omitted from this edition.

Besides making the changes mentioned above, the author has made many minor improvements ; he has clearly reconsidered the wording of practically every sentence in the book.

It should perhaps be made clear that, in spite of its title, the book deals with many topological questions that are not concerned merely with sets of points in the plane. This new edition will be even more useful than the first one, especially to beginners in topology. F. SMITHIES.

An Introduction to the Theory of Control in Mechanical Engineering. By R. H. MACMILLAN. Pp. xiv, 195. 30s. 1951. (Cambridge University Press)

The first book on servomechanisms to be produced in this country, which was written by Porter and reviewed in the *Gazette* (Vol. XXXV, No. 314), was followed shortly afterwards by the present volume. As his title implies, Macmillan covers a somewhat wider field than Porter, but since most control systems make use of some form of feed-back, actual or inherent, this extension may be more apparent than real. If a comparison between the two books is to be made, it will show Macmillan amplifying by means of numerous practical applications the general principles which Porter sets out very clearly, but with some brevity. The two books can therefore be regarded to a certain extent as complementary.

A valuable feature of the present book is the careful account of the various basic mechanical and electrical components with which control systems are built up. The inclusion of the word " mechanical " in the title indicates the emphasis of the book, which is shewn by the detailed descriptions and careful drawings of the various types of hydraulic motors, pumps and valves that are given. This is a deliberate reaction to a current tendency in the literature of the subject. Nevertheless the equivalent electrical elements are by no means neglected.

The first chapter is devoted to the general principles of control and gives definitions of some basic ideas, such as open and closed loops. In particular, control systems are classified into three types : regulators, process controls and remote position controls. Of special interest is the definition of a servo system. This has been the subject of much discussion and many alternative versions have been produced. These however all have two points in common ; firstly

the existence of a source of power and secondly the fact that this power is controlled or monitored by a feed-back of its output. The present author takes the unusual line of suggesting as a definition that a servo system is a control system of a type which can become unstable, arguing that instability is not possible in an open chain system, nor can it occur without a power supply. There seems to be a flaw in the logic of this ingenious reasoning, especially as one of the requirements of a servo is that it should be amply stable in all conditions of operation for which it is designed.

In the second chapter descriptions are given of various types of control system elements, including hydraulic pumps and motors, measuring elements, data transmission links, mechanical and electrical power amplifiers or relays and mechanical variable speed gears. The idea of a transfer function is introduced and its derivation for the above elements is explained in some detail. Transfer functions are expressed as functions of the operator $D = d/dt$, with the generality that this implies—a point to be commended. In the opinion of the reviewer this is an important and useful chapter, particularly to a reader whose main interest is in general principles, but who wishes to learn a little about the characteristics of individual elements of particular systems.

The elementary ideas of the theory of servos are introduced in the third chapter by means of specific examples. There is a section on manually controlled systems, with particular reference to the problem of tracking a target in anti-aircraft artillery. The chapter is concluded with a short section on discontinuous and non-linear systems. This, in the space available, can no more than touch upon one or two aspects of a large and fast developing branch of the subject. A linear system is here (and elsewhere) defined as one governed by a linear differential equation with constant coefficients. This of course is a matter of convention, but this definition does exclude a number of types of system in which the superposition principle (which is associated generally with the idea of linearity) is applicable. Numbered among these types are systems with time-variant parameters (sometimes called parametric forcing) and systems using sampled data—the latter providing a problem which is being studied extensively at the present time. It may even be argued that with this definition a finite time delay renders a system non-linear, unless one is prepared to consider differential equations of infinite order.

Servo system performance is discussed in the next chapter. The ideas of static and dynamic errors, stability, and harmonic and unit function response are developed in considerable detail. This is followed by a chapter on general servo analysis, in which extensive use is made of operators.

The next two chapters are devoted to the two most widely used techniques for obtaining and discussing the response of servo systems. The first of these is the Laplace transformation and the second the use of the various types of harmonic response diagram. In the latter chapter full accounts are given of the properties of the Nyquist diagram, the modified form of Whiteley and Marcy and the graphs showing attenuation and phase as functions of the logarithm of frequency. The volume is concluded with a short but interesting historical summary, a bibliography, and appendices on complex numbers and the Laplace transformation.

This book has been written primarily for engineers and its merits in this role will no doubt be discussed elsewhere. Nevertheless the mathematician has an important part to play in this field, both in research and in education, and it is appropriate here to offer some comments from his point of view. Three objectionable features are the use of the word "differential" for "derivative", the expression of the characteristic equation of a system as a function of D, which is an operator and not a number, and carelessness in

writing down operational formulae, for example $\frac{Dq_2}{k_2+AD^2}$ instead of $\frac{D}{k_2+AD^2}q_2$ and $Y_0(D)=\theta_0/\theta_i$ instead of $\theta_0=Y_0(D)\theta_i$. These are perhaps points for the purist. More serious is the author's statement of the stability criteria. In this it appears that Hurwitz generalised the conditions given by Routh, whereas in fact both of these authors gave general criteria in different but equivalent forms. Furthermore the Hurwitz rule is here stated incorrectly.

The method of the Laplace transformation is included, so the author says, with some reluctance and it must be stated that the treatment he gives is not such as to bring out its special merits. In addition there are a number of inaccuracies, a curious one of which is a reference to " Faltung's theorem ". This name refers of course to a process (convolution) and not to its discoverer.

Considered as a whole, this book could have been compressed into appreciably smaller space without sacrificing any of its substance. In particular, the general equations of a servo could have been introduced earlier, thus showing up more clearly the problems that are involved.

So far the comment we have offered is mainly of a critical nature, and it is by no means exhaustive in this respect. Nevertheless on the other side it can be said that the author has studied his subject extensively and has devoted considerable care to its exposition. He has many ideas that are both definite and sound, and much of his material should be of considerable value. The book appears in the distinguished livery of a Cambridge Press quarto volume and, as may be expected, the production is first class. There are numerous diagrams, well drawn and reproduced, although some of these have inadequate explanation. Finally, a valuable feature is the inclusion at the end of each chapter of a set of well chosen exercises for the student. B. M. BROWN.

Linear Transformations in n-dimensional Vector Space. An Introduction to the Theory of Hilbert Space. By H. L. HAMBURGER and M. E. GRIMSHAW. Pp. x, 195. 25s. 1951. (Cambridge University Press)

This book gives a very readable account of the theory, particularly of the reduction to canonical forms, of linear operators in n-dimensional linear spaces over the complex field. As the title indicates, it is intended to act as an introduction to the theory of operators in Hilbert Space, and hence the authors have, on the whole, confined themselves to the use of methods which can be carried over to Hilbert Space. The essence of these methods is that they are as far as possible geometrical, operating with subspaces and linear operators rather than with determinants and with the explicit matrix expression of the linear operators. In this they resemble the methods which are finding favour with modern algebraists : and they could be carried over with little change to discussion of linear spaces over an arbitrary algebraically complete field, though they would not be suitable for a discussion of, say, the reduction of real matrices to canonical forms by real transformations.

The first two chapters define n-dimensional vector space and the main types of operator. The vectors are defined concretely, as sets of n numbers, and the scalar product is introduced and plays a considerable part in the theory—its use results in particular in a presentation of the theory of the solution of linear equations which beginners are likely to find rather easier than the usual determinant-free discussions. The third chapter discusses the spectral representation of hermitian transformations, using for the discussion the variational method, the definition of eigenvalues by maximal properties. This is followed by a discussion of inequalities for eigenvalues of hermitian operators which readers will find very valuable : it includes an account of the recent inequalities of Aronszajn, which are of importance in the practical estimation of eigenvalues. The functional calculus for hermitian operators is also discussed.

The fourth chapter is one of the most interesting in the book : it discusses the theory of the general linear operator by methods which the authors have worked out very carefully ; methods in which the power of the geometrical treatment emerges. These methods have been applied by the senior author to a class of operators in Hilbert Space : but there still does not exist a complete theory for operators other than the normal operators, and there is certainly much yet to be done in the carrying over of the ideas of this chapter to infinite dimensional spaces. The Jordan canonical form and the theory of elementary divisors are fully discussed. The last chapter deals with spaces in which a positive-definite Hermitian form is defined—in other terms, with the canonical form of a pencil of operators one of which is positive definite ; and the applications to the theory of small oscillations of dynamical systems are discussed. The book concludes with an excellent set of historical notes, a comprehensive bibliography, and an index.

The authors have not allowed themselves to be bound too firmly by the intention expressed by their subtitle : they have at times made certain compromises with the evident aim of producing a richer and more readable account of n-dimensional space. This is seen most clearly in the third chapter : the variational methods there used cannot be applied to all Hilbert Space operators ; but their use has the advantage of introducing the reader to ideas which are of great practical importance wherever they can be applied. More debatable points are the use made of the scalar product, and the definition of the space as a concrete space of sets of n numbers. The use of the scalar product has great technical advantages in calculating with subspaces, and is valuable because of its importance in Hilbert space ; but its use here may disguise the fact that the theory expounded is true in affine, not merely in metric spaces. This point could easily be cleared up by explaining that an n-dimensional affine space is isomorphic with its dual. The definition of the vector space, and of the scalar product, by abstract arguments would have been a good introduction to the abstract definition of Hilbert Space, and would have saved a certain amount of repetition in the last chapter ; however, the line the authors have chosen has advantages for less mature readers, and the treatment does not make much use of the coordinates.

It can be said without hesitation that this is an excellently written account of its subject, and one that can confidently be recommended. We look forward to the authors' work on Hilbert Space.

J. L. B. C.

Leçons d'Analyse Fonctionnelle. By FREDERIC RIESZ and BELA SZ. NAGY. Pp. viii, 448. $7.70. 1952. (Hungarian Academy of Sciences)

Functional Analysis, and particularly the theory of linear spaces with which this book deals, has a large and growing literature ; but in spite of the growing importance and diversity of its applications, it has lacked readable accounts, suitable for student as well as specialist, of the subject as a whole. For a major part, the part of most frequent application, this need is met brilliantly by the book under review.

The authors' names are sufficient guarantee of the excellence of the work. Both have made major contributions to the subject and have written important and, what is nowadays rare, lucid books on it. The elder, Riesz, is known to all analysts ; his work began in the days when Lebesgue's integral and Hilbert's work on Integral Equations transformed analysis : he contributed greatly to both theories, simplifying their formulation and acting as a pioneer in giving them their modern, more abstract, form. A search for the general ideas underlying theories and for a classical elegance of exposition have been characteristic of his work. These tendencies have done much to form the present book.

It begins with a discussion of the differentiation of monotonic functions, developed with no use of the theory of integration, and then gives an account of the Lebesgue integral by a definition based on extension from step functions to wider classes of functions : a definition which is frequently preferred to that of Lebesgue and is due to Riesz. The L^p spaces are discussed as metric linear spaces, and other definitions of the Lebesgue integral, including that of Lebesgue, are given. The Stieltjes integral is then defined, directly, and as a functional on the space of continuous functions ; its extension to the Lebesgue-Stieltjes integral is discussed. An account of the Daniell abstract integral closes the first third of the book, which forms a selfcontained account of real variable theory.

The second part begins by discussing integral equations, first of all by more or less classical methods and then by means of the abstract theory of completely continuous operators on L^2. Abstract Hilbert Space and Banach Spaces are defined in the next chapter : most of the arguments are carried through for Hilbert Space, but the possible generalisations to Banach Spaces are pointed out. Four chapters are devoted to the theory of symmetric and normal operators in Hilbert Space, going with increasing generality from completely continuous to bounded and then to unbounded functions. The last of these chapters concerns itself with the functional calculus for these operators and with perturbation theory. The major applications to ordinary analysis, the boundary problems of potential theory, other problems of differential equations, almost periodic functions, are dealt with quite fully. The next chapter discusses the theory of groups and semigroups of transformations—dealing mainly with the unitary and self-adjoint representations of the group of real numbers, but indicating the theory for other groups : and it concludes with an account of ergodic theory. The final chapter deals with the spectral theory of general bounded operators. It commences with an account of the modern developments of the use of complex integration for the problem, which includes a very interesting application to Wiener's theorem concerning absolutely convergent Fourier Series, and proceeds to some recent contributions of von Neumann.

The book is so written that the ordinary honours course in analysis will suffice for its reading. Its method of exposition is to prepare the way for abstract concepts by well chosen examples of concrete cases. Generality for its own sake has been avoided ; greater generality might sometimes have increased the value of the work as a book of reference, but the authors have made a good compromise between this and the very great pedagogic advantages of teaching the general methods by their application to the concrete cases. As a work of exposition this deserves to rank with the great French classics. It can be recommended strongly to all students of analysis ; and its readers will share the authors' gratitude to the Hungarian Academy of Sciences for making it available to an international public.

The book is well and accurately produced, and closes with a comprehensive bibliography and a useful index. There seem to be few errors worth mentioning, but the following may be :— the reference to the paper of E. Heinz (see P. 369) is omitted : it is *Math. Ann.* 123, (1951), 415–438 ; and the statement on P. 214 that it is not known whether the Fredholm radius of a completely continuous operator in a Banach space is always infinite is no longer true : the result has been proved by Nikolsky. J. L. B. C.

Automatic and Manual Control. Papers contributed to the Conference at Cranfield, 1951. Pp. xi, 584. 50s. 1952. (Butterworth)

This volume of nearly 600 pages, well printed on excellent paper, contains the report on the Cranfield conference on automatic control, organised by the

Department of Scientific and Industrial Research in July 1951. As Sir Ben Lockspeiser said in his presidential address, " the conception of automatic control arises basically from the use of devices which measure the departure of the physical state of a system from a desired normal, and the utilisation of this information to restore the system to the normal state." In recent years this characteristic has been recognised as a common feature in engineering, in the manufacture of chemical products, in aeronautical control and in weapon control as well as in other servo-systems. Professor N. Wiener has created Cybernetics, a new branch of science (or of philosophy), combining various aspects which contribute to control in industry and in society. This is not mentioned in the book under review, but it is apparent that the boundaries of the new field have not precisely been traced as yet. There is, for instance, a section on the Analysis of the Behaviour of Economic Systems, which gives an outline of a lecture by R. Stone (printed, in extenso, in an economic periodical). The reviewer feels that the connection with automatic control is purely formal and that any analogies derived therefrom are hardly illuminating.

The contributors to the Cranfield conference were, in the first place, speaking or writing as physicists and engineers and thus the interests of sociologists or historians of science were perforce kept in the background. After a preface, an introduction and the presidential address, there are nine sections, the largest of which are headed " General Theory " and " Non-linear Problems ". Process control in the chemical industry, the human operator, and step by step servos are also dealt with. At the end of the book we find one page of advice on where the theory of complex variables could be looked up. Possibly the editor, Professor Tustin, felt that too much mathematical knowledge had been expected by some authors. As one would expect in a book of this type, the papers do not contain much explanation of the basic mathematical techniques which are being used, or referred to, but there is one paper, by B. M. Brown, on " Application of Finite Difference Operators to Linear Systems ".

The book will be useful to students and other readers who wish to extend their knowledge of this field beyond their immediate concern. On the whole, however, it is fair to be warned in the introduction that " the papers in general are written on the presumption that the reader has some prior knowledge of the theory of control systems." An author index and a subject index at the end of the book help to enhance the value of it as a source of reference. The report contains also the discussions during the conference, sometimes after the last of a group of related papers. It is clear from a perusal of the volume that the papers were stimulating, the response of the audience keen, and that thus the conference has achieved its main object. S. V.

The Algebra of Vectors and Matrices. By THOMAS L. WADE. Pp. ix, 189. $4.50. 1951. (Addison-Wesley Press Inc., Cambridge, Mass.)

This is a pleasant and very elementary introduction to linear algebra. It is intended not so much for the mathematical specialist but for the Applied Scientist who is interested in the applications of this modern branch of algebra to his own field—be it physics, engineering, theoretical chemistry, statistics, multiple factor analysis, biometrics or psychometrics. The book only leads the reader up to the place where such applications become possible, but a bibliography shows the way to further study. As a text-book for Honours undergraduates in our Universities it is probably less suitable, firstly because it does not go far enough, and secondly because in a number of places the author does not actually prove his results but is content to state " It can be proved . . ." without giving chapter and verse and then goes on using the results.

The mathematical background required is very slight: a knowledge of simple coordinate geometry and the usual theorems and manipulations of determinants. The author's treatment is cursory and descriptive rather than abstract and he does not aim naturally at the greatest possible generality. The first chapter introduces the basic concepts of modern algebra—groups, rings, integral domains, and fields; the second deals with two- and three-dimensional vectors (the conceptual difficulties connected with free and localised vectors are not mentioned), the third gives geometrical applications and the preliminary treatment of linear dependence. This is continued in the next chapter on vectors in n dimensions. The following six chapters, the major part of the book, deal with matrix theory. They are clearly written and give many illustrative examples. The Cayley-Hamilton theorem is proved and the minimum polynomial is briefly mentioned, but not actually established. The canonical form is only treated in the case of distinct characteristic roots which may evoke in the inexperienced reader a distorted picture of simplicity. An unusual feature is that the general theory of linear equations comes right towards the end of the book. This simplifies the proofs, but uses a very heavy apparatus of previous theorems. K. A. H.

Calculus and Analytic Geometry. By G. B. THOMAS, Jr. Pp. 685. $6. 1951. (Addison-Wesley, Cambridge, Mass.)

This book is intended for students who already possess a knowledge of algebra, plane and solid geometry, and trigonometry. After reading it, they will be well prepared for the study of differential equations and the advanced calculus.

The first five chapters are devoted to introducing the ideas of differentiation and integration and to applying these new methods to practical problems. After introducing conic sections and polar coordinates, more applications of the calculus follow and methods of integration are studied in detail. Subjects dealt with in the end portion of the book include solid geometry and vectors, partial differentiation, multiple integrals, infinite series, and complex numbers.

The book has been written so that certain chapters need not be read in full during a first reading, for example, the chapter on determinants. Determinants of orders 2 and 3 are, in fact, all that are needed in the remainder of the book.

Instead of being set in type the book has been prepared on an automatic typewriter in order to allow the author complete freedom to revise the book prior to setting it in type for the next edition. Consequently this final edition will be less bulky than the present volume. The book in its present form is, however, neatly and clearly set out and is easily readable. The diagrams are well drawn and those concerned with three dimensions are particularly good, shading being used to give the idea of depth. There are a great many worked examples in the text and numerous problems for the student to solve.

P. M. H.

"Godfrey and Siddons." Pp. 16, two plates. 1952. (Cambridge University Press)

To commemorate the fiftieth anniversary of the publication by the Cambridge Press of *Elementary Geometry*, this pamphlet describes, often in Siddons' own words, the way in which the famous collaboration began, thought, wrote; and, almost incidentally, it describes the main steps in the reform of mathematical teaching which has been so prominent a feature of school progress since 1900. The Mathematical Association may well take pride in this jubilee, for the Teaching Committee has repaid the inspiration which it received from Godfrey and Siddons by steady and enthusiastic support for their teaching ideals. T. A. A. B.

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